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PARTIAL DIFFERENTIAL EQUATIONS

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SMAM23: PARTIAL DIFFERENTIAL EQUATIONS

SYLLABUS

Unit I

Mathematical Models and Classification of second order equation: Classical equations - Vibrating string – Vibrating membrane – waves in elastic medium – Conduction of heat in solids – Gravitational potential – Second order equations in two independent variables – canonical forms – equations with constant coefficients – general solution.

Chapter 1: Sections 1.1 to 1.10

Unit II

Cauchy Problem: The Cauchy problem – Cauchy-Kowalewsky theorem – Homogeneous wave equation – Initial Boundary value problem- Non-homogeneous boundary conditions – Finite string with fixed ends – Non-homogeneous wave equation – Riemann method – Goursat problem – spherical wave equation – cylindrical wave equation.

Chapter 2: Sections 2.1 to 2.11

UNIT III

Method of separation of variables: Separation of variable- Vibrating string problem – Existence and uniqueness of solution of vibrating string problem - Heat conduction problem – Existence and uniqueness of solution of heat conduction problem – Laplace and beam equations

Chapter 3: Sections 3.1 to 3.6

Unit IV

Boundary Value Problems: Boundary value problems – Maximum and minimum principles – Uniqueness and continuity theorem – Dirichlet Problem for a circle, a circular annulus, a rectangle – Dirichlet problem involving Poisson equation – Neumann problem for a circle and a rectangle.

Chapter 4: Sections 4.1 to 4.9

Unit V

Green's Function: The Delta function – Green's function – Method of Green's function – Dirichlet Problem for the Laplace and Helmholtz operators – Method of images and Eigen functions – Higher dimensional problem – Neumann Problem.

Chapter 5: Section 5.1 to 5.9

Text Book

1.TynMyint-U and Lokenath Debnath, *Partial Differential Equations for Scientists and Engineers* (Third Edition), North Hollan, New York, 1987

SMAM23: PARTIAL DIFFERENTIAL EQUATIONS

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Unit I

Mathematical Models and Classification of second order equation: Classical Equations-Vibrating string – Vibrating membrane – waves in elastic medium – Conduction of heat in solids – Gravitational potential – Second order equations in two independent variables – canonical forms – equations with constant coefficients – general solution.

Chapter 1: Sections 1.1 to 1.10

Mathematical Models:

1.1.The Classical Equations:

The three basic types of second order partial differential equations are:

a) The wave equation $u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0$ ....(1)

b) The heat equation $u_t - k(u_{xx} + u_{yy} + u_{zz}) = 0 \ldots \ldots . . (2)$

c) The Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$ ....(3)

- We list a few more common linear partial differential equations.

d) The Poisson equation$\nabla^2 u = f(x, y, z)$ ....(4)

e) The Helm holt z equation $\nabla^2 u + \lambda u = 0 \quad \ldots \ldots \ldots \ldots (5)$

f) The biharmonic equation $\nabla^4 u = \nabla^2(\nabla^2 u) = 0 \ldots \ldots \ldots \ldots \ldots (6)$

g)The biharmonic wave equation $u_{tt} + c^2 \nabla^4 u = 0 \ldots \ldots \ldots \; (7)$

h) The telegraph equation $u_{tt} + au_t + bu = c^2 u_{xx} \; \ldots \ldots . . (8)$

(i) The Schrodinger equation in Quantum physics

- Schrodinger time dependent wave equation we know that $\lambda = \frac{h}{mu}$

$$x = \frac{h}{px}$$

$$p_x = \frac{h}{\lambda} \times \frac{2\pi}{2\pi}$$

$$= \frac{h}{2\pi} \cdot \frac{2\pi}{\lambda}$$

$Px = \hbar \frac{2\pi}{\lambda}$ where $\hbar = \frac{h}{2\pi}$

$P_x = \hbar \cdot k$ where $k = \frac{2\pi}{\lambda}$

$k = \frac{p_x}{\hbar} \ldots \ldots \ldots . (1)$

By Quantum theory.

$$E = hv \times \frac{2\pi}{2\pi}$$
$$= \frac{h}{2\pi} \times 2\pi v$$
$$= \hbar 2\pi v$$
$$E = \hbar\omega \qquad (\omega = 2\pi v)$$
$$\omega = \frac{E}{\mathrm{h}} \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

Total Energy of system

$$E = k \cdot E + P \cdot E$$
$$E = \frac{1}{2}mv^2 + v$$
$$E = \frac{1}{2}\frac{mmv^2}{m} + v$$
$$= \frac{1}{2}\frac{m^2v^2}{m} + v$$
$$= \frac{1}{2}\frac{Px^2}{m} + v \; (\because P_x = mv)$$

Multiple by ψ $\psi E = \frac{1}{2}\frac{px^2}{m}\psi + v\psi$ ………….(3)

plane wave equation

$$\psi = ae^{-i\left[\frac{E}{\hbar}t - \frac{px}{\hbar}x\right]}$$
$$\psi = ae^{i\left[\frac{px}{\hbar}x - \frac{E}{\hbar}t\right]}$$
$$\psi = e^{i/\hbar}[xP_x - Et] \qquad \ldots\ldots\ldots (4)$$

Differentiation (4) w.r.to t $\frac{d\psi}{dt} = ae^{i/\hbar[xP_x - Et]}\left(-E^{i/\hbar}\right)$

$$\frac{d\psi}{dt} = -E\psi\frac{i}{\hbar} \quad \text{(from (4))}$$
$$= -E\frac{\psi i, i}{i\hbar}$$
$$\frac{d\psi}{dt} = \frac{E\psi}{i\hbar}$$
$$i\hbar\frac{d\psi}{dt} = E\psi \ldots\ldots\ldots.(5)$$

Diff (4) w.r.to x

$$\frac{d\psi}{dx} = ae^{i/\hbar[xP_x - Et]}P_x^{i/\hbar}$$
$$\frac{d\psi}{dx} = \frac{\psi P_x i}{\hbar} \ldots\ldots\ldots (6)$$

Differentiation (6) w.r.to x

$$\frac{d^2\psi}{dx^2} = \frac{d\psi}{dx} p_x^{i/\hbar}$$

$$= \psi p_x \frac{i}{\hbar} \cdot p_x{}^{i/\hbar} \quad \text{from (6)}$$

$$\frac{d^2\psi}{dx^2} = -\psi \frac{p_x^2}{\hbar^2}$$

$$-\hbar^2 \frac{d^2\psi}{dx^2} = \psi p_{x^2} \quad \ldots\ldots\ldots\ldots (7)$$

(from (6) (3) becomes

$$i\hbar \frac{d\psi}{dt} = \frac{1}{2m}\left(-\hbar^2 \frac{d^2\psi}{dx^2}\right) + v\psi \quad \ldots\ldots\ldots..(8) \quad [\text{ from } (5) \times (7)]$$

for 3 dimension

$$i\hbar\psi_1 = \left[\frac{-\hbar^2}{2m} \nabla^2\psi + v(x,y,z)\psi\right]$$

$$i\hbar\psi_1 = \left[\frac{-\hbar^2}{2m} \nabla^2 + v(x,y,z)\right]\psi \quad \ldots\ldots\ldots\ldots. (9)$$

This equation is the Schrodinger time dependent wave equation

- Schrodinger time independent wave equation

$$W \cdot K \cdot T \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \ldots\ldots\ldots\ldots (1)$$

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \quad \ldots\ldots\ldots\ldots..(2) \qquad \left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

Solution for equation (2),

$$\psi(x,y,z) = \psi_0(x,y,z)e^{-i\omega t} \quad \ldots\ldots\ldots\ldots\ldots.(3)$$

Diff (3) $\omega \cdot r \cdot t$

$$\frac{\partial\psi}{\partial t} = -i\omega e^{-i\omega t}\psi_0(x,y,z)$$

Again Differentiation w.r.to t

$$\frac{\partial^2\psi}{\partial t^2} = -i\omega(-i\omega)e^{-i\omega t}\psi_0(x,y,z)$$

$$= -\omega^2 e^{-i\omega t}\psi_0(x,y,z)$$

$$\frac{\partial^2\psi}{\partial t^2} = \omega^2\psi \quad \ldots\ldots\ldots\ldots (4) \quad (by\ (3))$$

sub (4) in (2)

$$\nabla^2\psi = \frac{1}{v^2}(-w^2\psi)$$

$$\nabla^2\psi + \frac{w^2\psi}{v^2} = 0 \quad \ldots\ldots\ldots\ldots (5)$$

We know that,

$$\pi\ \omega = 2\pi\nu$$

$$\omega = 2\pi(v/\lambda)$$

$$\frac{\omega}{v} = \frac{2\pi}{\lambda}$$

$$\frac{\omega^2}{v^2} = \frac{4\pi^2}{\lambda^2} \quad \ldots\ldots\ldots\ldots (6)$$

sub (6) in (5) we get

$$\nabla^2\psi + \frac{4\pi^2}{\lambda^2}\psi = 0$$

$$\nabla^2\psi + \frac{4\pi^2\psi m^2 v^2}{h^2} = 0 \quad \ldots\ldots\ldots..(7) \quad \left(\because \lambda = \frac{h}{mv}\right)$$

$$E = K\cdot E + P\cdot E$$

$$= \frac{1}{2}mv^2 + v$$

$$E - v = \frac{1}{2}mv^2$$

$$2(E - v) = mv^2$$

$$2m(E - v) = m^2v^2 \ldots\ldots\ldots\ldots (8)$$

sub(8) in (7) we get

$$\nabla^2\psi + \frac{4\pi^2\psi}{h^2}[2m(E - v)] = 0$$

$$\nabla^2\psi + \frac{2m\psi}{h^2/4\pi^2}(E - v) = 0$$

$$\nabla^2\psi + \frac{2m(E - v)}{\hbar^2}\psi = 0$$

$$\nabla^2\psi + \frac{2m}{\hbar^2}\big(E - v(x, y, z)\big)\psi = 0 \ldots\ldots\ldots\ldots (9)$$

This is the Schrodinger time independent wave equation.

j) The Klein - Gordon Equation

Schrodinger's relativistic wave equation. The non-relativistic Schrodinger equation obtained by replacing P by i th $-i\hbar\nabla$ and E by $i\hbar\,\partial/\,\partial t$ in the classical energy expression of a free particle $E = P^2/2m$ & allowing the resulting operator equation on the wave equations

The corresponding relativistic energy relation

$E^2 = c^2p^2 + m^2c^4$ ... ... (1)

By replacing $E\&P$ we get

$$-\hbar^2 \frac{\partial^2}{\partial t^2} = -c^2\hbar^2\nabla^2 + m^2c^4x$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace equation.

operator in rectangular Cartesian co-ordinate x, y, z Allowing this operator equation to operate on the wave function $u(r, t)$

wave function

$$* u - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u = -\nabla^2 u + \frac{m^2c^2}{\hbar^2}u$$

$$\frac{-1}{c^2}\frac{\partial^2 u}{\partial t^2} + \nabla^2 u = \frac{m^2c^2}{\hbar^2}u$$

$$\Rightarrow \left[\nabla^2 - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\right]u = \frac{m^2c^2}{\hbar^2}u$$

$\square\, u = \lambda^2 u$ where $\lambda = \frac{m^2c}{\hbar}$ and

$$\sigma = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

is the D' Alembertion and all the equation λ, a, b, c, m, E are constants and $h = 2m\hbar$ is the "plank constant".

1. 2. The vibrating string:

Vibrations of a stretched string (or) Derive one dimensional wake equation.

We shall obtain the equation of the motion of the string under the following assumptions.

1. The string is flexible and elastic, that is, the string cannot resist bending moment and thus the tension in the String is always in the direction of the tangent to the existing profile of the string.
2. There is no elongation of the single segment of the string and hence by Hooke's law, the tension is constant.

   (i.e.) string move only in the vertical direction, there is no motion in the horizontal direction.

   (i.e.) sum of the forces in the horizontal direction must be zero

   $\therefore T\cos\beta - T\cos\alpha = 0$

$T\cos\beta = T\cos\alpha = T$ (constant say) ……... (1)

3. The weight of the string is small compared with the tension in the string.
4. The direction is small compared with the length of the string.
5. The slope of the displaced string at any point is small compared with unity.
6. There is only pure transverse vibration.

(we consider a differential element of the string.

Let T be the tension at the end point. The force acting on the element of the string in the vertical direction are $T\sin\beta - T\sin\alpha$

Gravitational forces on the string is neglected,

Then by Newton's and law of motion,

Mass x acceleration = resultant force

$\Rightarrow \rho\Delta s\cdot\left(\frac{\partial^2 u}{\partial t^2}\right) = \tau\sin\beta - \tau\sin\alpha$ ……. (1)

Where ρ is the line density & Δs is the small arc length of the string. Since the slope of the displaced string is small, we have $\Delta s \cong \Delta x$ ... ... (2)

Since the angle $\alpha + \beta$ are small $\sin\alpha \simeq \tan\alpha,\ \sin\beta \simeq \tan\beta$ ……(3)

$\Rightarrow \frac{\rho}{T}\Delta x u_{tt} = \tan\beta - \tan\alpha \ldots\ldots\ldots (4)$

$$\Rightarrow u_{tt} = \frac{T}{\rho\Delta x}[\tan\beta - \tan\alpha]$$

By calculus

$\tan\alpha = (u_x)_x \rightarrow$ (s) (b)pe of the string at x) ………… (5)

$\tan\beta = (U_x)x + \Delta x$ (slop of the string at $x + \Delta x$. ……….. (6)

$$\therefore\ (2)\ \Rightarrow u_{tt} = \frac{T}{\rho\Delta x}[(u_x)_{x+\Delta x} - (u_x)_x]$$

$$= \frac{T}{\rho}\left[\frac{(u_x)_{x+\Delta x} - (U_x)_x}{\Delta x}\right]$$

$$= \frac{T}{\rho}U_{xx} \text{ at } \Delta x \rightarrow 0 \qquad \therefore \lim_{\Delta x\to 0}\frac{(U_x)x + \Delta x}{\Delta x} = u_{xx}$$

$$\Rightarrow u_{tt} = c^2 u_{xx} \text{ where } c^2 = T/\rho \rightarrow (7)$$

This is called the one-dimensional wave equation)

Note: If there is an external force f per unit length acting on the string (3) assumes the form

$u_{tt} = c^2 u_{xx} + f^*, f^* = f/p \rightarrow$ (4)

Where f may be pressure, gravitation, resistance and so on...

1.3. The vibrating membrane:

Derivation of two dimensional wave equation We shall obtain the equation for the vibrating membrane under the following assumption.

1. The membrane is flexible and elastic.
2. The Tension is constant.
3. The weight of the membrane as small as compared with the tension in the membrane.
4. The deflection is small compared with the minimal diameter of the membrane.
5. The slope of the displayed membrane at any point is small compared with unity.
6. There is only pure transverse vibration.

consider a small element of the membrane. Since the defection and slope are small, the area of the elemental approximately equal to $\Delta x \Delta y$.

If T is the tensile force per unit length, then the forces acting on the sides of the element are $T\Delta x$ & $\text{T}\Delta y$.

$\therefore$ The forces acting on the element of the membrane in the versicle direction are

$$(T\Delta x \sin\beta - T\Delta x \sin\alpha) + (T\Delta y \sin\delta - T\Delta y \sin\gamma) \qquad \ldots\ldots\ldots (1)$$

since the slopes are small,

$$\sin\alpha = \tan\alpha, \sin\beta = \tan\beta, \sin\nu = \tan\gamma,$$
$$\sin\delta = \tan\delta \ldots\ldots\ldots.(2)$$

$$(1) \Rightarrow T\Delta x\tan\beta - T\Delta x\tan\beta + T\Delta y\tan\delta - T\Delta y\tan\gamma$$
$$\Rightarrow T\Delta x(\tan\beta - \tan\alpha) + T\Delta y(\tan\delta - \tan\gamma) \ldots\ldots\ldots (3)$$

Now, by newton's 2^{nd} law of motion, the resultant force is equals to the mass times the acceleration.

hence $T\Delta x(\tan\beta - \tan\alpha) + T\Delta y(\tan\delta - \tan\nu) = \rho\Delta\text{Autt} \ldots\ldots\ldots\ldots (4)$

Where ρ is the man per unit area.

$\Delta A = \Delta x\Delta y$ is the area of this element.

$u(x, y, t)$ is the position of the membrane at time ' t ' after an interval disturbance is give

x is the displacement along x axis

y is the displacement along y axis

Now, $\tan\alpha =$ slope of membrane at $x_1 = u_y(x_1, y)\text{t}\}$

$\tan\beta =$ slope of membrane at $x_2 = U_y(x_2, y+\Delta y)t$

$\tan y =$ slope of membraneat $y_1 = u_x(x, y_1)t$ .... (5)

$\tan\delta =$ slope of membrane at $y_2 \Rightarrow$ at $y_2 = U_x(x+\Delta x, y_2)t$

Where $x_1 \times x_2$ are values of x between x & $x + sx$, y_1 & y_2 are values of y between y & $y + \Delta y$ sub there values in (4) we get

$$T\Delta x\left[u_y(x_2, y+\Delta y) - u_y(x_1, y)\right] + T\Delta y[u_x(x+\Delta x, y_2)$$
$$-U_x(x, y,)] = \rho\,\Delta A\, U_{tt}$$
$$\Rightarrow T\Delta x\left[u_y(x_2, y+\Delta y) - u_y(x_1, y)\right] + T\Delta y[u_x(x+\Delta x, y_2)$$
$$-u_x(x, y_1)] = \rho\,\Delta x\,\Delta y\, u_{tt} \ldots\ldots\ldots\ldots(6)$$

$$\Rightarrow \frac{T}{\rho}\left[\frac{u_y(x_2, y+\Delta y) - u_y(x, y)}{\Delta y} + \frac{u_x(x+\Delta x, y_2) - u_x(x_1, y_1)}{\Delta x}\right] = u_{tt} \quad \ldots\ldots\ldots(7)$$

$Taking\ limit\ in\ (7) as\ \Delta x \to 0, \quad \Delta y \to 0\ we\ get,$

$$\frac{T}{\delta}\left[u_{yy} + u_{xx}\right] = u_{tt}$$

$$\therefore U_{tt} = c^2\left(u_{yy} + U_{xx}\right) \quad \ldots\ldots\ldots\ldots(8)\ , where\ c^2 = \frac{T}{\delta}$$

This equation is called the two dimensional wave equation

Note: If there is an external force f per unit area acting on the membrane equation (8) take the form $U_{tt} = f^2\left(U_{yy} + U_{xx}\right) + f^*$ where $f^* = f/\delta$

1.4. Waves in an Elastic Medium:

Derivation of three dimensional wave equation:

If a small disturbance is originated at a point in an elastic medium, neighbouring particles are set into motion and the medium is put under a state or stain

we consider such states of motion to extend in all directions. We assume that the displacement of the medium is small.

Let the body under investigation be homogenous and isotropic.

Let Δv be a differential volume of the body and let the stresses acting on the forces of the volume be $\tau_{xx}, \tau_{yy}, \tau_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{zy}, \tau_{yz}, \tau_{zx}$. The first three stress are called normal stresses and the rest are shear stresses.

We shall assume that the stress tensor $\tau_{ij} = \tau_{ji}$, *i,j=x,y,z.*

$\therefore \tau_{xy} = \tau_{yx}; \tau_{zx} = \tau_{xz}; \tau_{yz} = \tau_{zy}$

Neglecting the body forces, the sum of all the forces acting on the volume elements in the

x-direction is

$$[(\tau xx)_{x+\Delta x} - (\tau xx)x]\Delta y\Delta z + [(\tau xy)_{x+\Delta x} - (\tau xy))_x]\Delta x\Delta z$$

$$+ [(\tau xz)_{x+\Delta x} - (\tau xz)x]\Delta x\Delta y \ldots\ldots\ldots (1)$$

By Newton law of motion

Resultant forces is equal to the mass times the acceleration. Thus we obtain

$$[(\tau xx)_{x+\Delta x} - (\tau xx)x]\Delta y\Delta z + [(\tau xy)_{x+\Delta x} - (\tau xy)x] \Delta x\Delta z + [(\tau xz)x + \Delta x - (\tau xz)x]\Delta x\Delta y = \rho\Delta x\Delta y\Delta z u_{tt}$$

[Where ρ is the densify of the body and w is the displacement component in x-direction]

$$\Rightarrow [(\tau_x)_{x+\Delta x} - (\tau_{xx})_x]\frac{\Delta x\Delta y\Delta z}{\Delta x} + [(\tau xy)x + \Delta x - (\tau xy)x]\frac{\Delta x\Delta y\Delta z}{\Delta y}$$
$$+[(\tau_{xz})x + \Delta x - (\tau xz)x]\frac{\Delta x\Delta y\Delta z}{\Delta z} = \rho\Delta x\Delta y\Delta z$$

$$\Rightarrow \left[\frac{(\tau_{xx})x+\Delta x - (\tau_{xx})x}{\Delta x}\right] + \left[\frac{(\tau_{xy})x+\Delta x - (\tau_{xy})x}{\Delta y}\right] + \left[\frac{(\tau_{xz})x+\Delta x - (\tau_{xz})x}{\Delta z}\right] = \rho u_{tt} \quad \ldots\ldots\ldots (2)$$

Taking limit $\Delta U \to 0$

we get

$$\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} = \rho\frac{\partial u^2}{\partial t^2} \quad \ldots\ldots\ldots\ldots (3)$$

Similarly, $\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} = \rho\frac{\partial^2 v}{\partial t^2}$ …………(4)

And $\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\tau_{zz}}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2}$ …………(5)

where v and w are the displacement component in y and z directions respectively.

Now, we define linear strain

$$\xi_{xx} = \frac{\partial u}{\partial x}, \ \xi_{yz} = \xi_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\xi_{yy} = \frac{\partial v}{\partial y} \ \xi_y = \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}$$

$$\xi zz = \frac{\partial\omega}{\partial z} \ \xi zx = \frac{\partial\omega}{\partial x} + \frac{\partial u}{\partial z} \quad \ldots\ldots\ldots (6)$$

In which $\xi_{xx}, \xi_{yy}, \xi_{zz}$ represent unit elongations and $\xi yz, \xi zx, \xi xy$ represent unit shearing strains

In the care of an isotropic body, generalized nook's low takes the form.

$$\begin{array}{ll} \tau_{xx} = \lambda\theta + 2\mu\xi_{xx} & \tau_{yz} = \mu\xi_{yz} \\ \tau_{yy} = \lambda\theta + 2\mu\xi_{yy} & \tau_{zx} = \mu\xi_{zx} \quad \ldots\ldots\ldots.. (7) \\ \tau_{zz} + \lambda\theta + 2\mu\xi_{zz} & \tau_{yx} = \mu\xi_{yx} \end{array}$$

where $\theta = \xi xx + \xi yy + \xi zz$ is called dilation and $\lambda k\mu$ are Lame's constants.

Expressing stresses in terms of displacements

$$\text{we have} \left.\begin{array}{l} \tau_{xx} = \lambda\theta + 2\mu\frac{\partial u}{\partial x} \\ \tau_{xy} = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] \\ \tau_{xz} = \mu\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\right] \end{array}\right\} \quad \ldots\ldots\ldots..(8)$$

Differentiating equation (8)

$$\frac{\partial\tau_{xx}}{\partial x} = \lambda\frac{\partial\theta}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial\tau_{xy}}{\partial x} = \mu\frac{\partial^2 v}{\partial x\,\partial z} + \mu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial\tau_{x^2}}{\partial x} = \mu\frac{\partial^2 u}{\partial x\,\partial z} + \mu\frac{\partial^2 u}{\partial z^2}$$

sub. (9) in (3) we get

$$\left[\lambda\frac{\partial\theta}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2}\right] + \left[\mu\frac{\partial^2 u}{\partial x\,\partial y} + \mu\frac{\partial^2 u}{\partial y^2}\right] + \left[\mu\frac{\partial^2 u}{\partial x\,\partial z} + \mu\frac{\partial^2 u}{\partial z^2}\right]$$

$$= \rho\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\lambda\,\partial\theta}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial x\,\partial y} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x\,\partial z}$$

$$+\mu\frac{\partial^2 y}{\partial z^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \lambda\frac{\partial\theta}{\partial x} + [\mu[\left[\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\Bigg] +$$

$$\mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial x\,\partial z}\right] = \rho\frac{\partial^2 u}{\partial t^2} \ldots\ldots\ldots.(10)$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial x\,\partial z} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z}\right] = \frac{\partial v}{\partial x}$$

$$\text{and } \Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The symbol Δ or ∇^2 is called the Laplace operator Hence (10) bennes

$$\lambda \frac{\partial \theta}{\partial x} + \mu \nabla^2 u + \mu \frac{\partial \theta}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow (\lambda + \mu)\frac{\partial \theta}{\partial x} + \mu \nabla^2 u = \rho \frac{\partial^2 u}{\partial t^2} \quad \ldots \ldots \ldots . . (11)$$

Similarly,

$$(\lambda + \mu)\frac{\partial \theta}{\partial x} + \mu \nabla^2 v = \rho \frac{\partial^2 v}{\partial t^2} \quad \ldots \ldots \ldots . . (12)$$

$$(\lambda + \mu)\frac{\partial \theta}{\partial x} + \mu \nabla^2 w = \rho \frac{\partial^2 w}{\partial t^2} \quad \ldots \ldots \ldots \ldots (13)$$

In vector form, $(\lambda + \mu)$ grad div $u + \mu \nabla^2 u = \rho u_{tt}$

where $u = ui + vj + wk$ and $\theta =$ div u

case (i)

$$(\lambda + \mu)\frac{\partial \theta}{\partial y} + \mu \nabla^2 v = \rho \frac{\partial^2 v}{\partial t^2}$$

If div u $= 0$, the general equation becomes

$$\mu \nabla^2 u = \rho utt$$

$$\Rightarrow utt = \frac{\mu}{\rho} \nabla^2 u$$

$$\Rightarrow \text{att} = c^2 \nabla^2 u \quad \ldots \ldots \ldots \ldots . (15)$$

where the velocity c of Propagated wave is $c = \sqrt{\mu/\rho}$

This is the case of an equivoluminal wave propagation, the volume expansion θ is zero for way moving with this velocity. Sometimes these waves are called waves of distortion because the velocity of propagation depends on $\mu \times \rho$, the shear modulus μ characterizes the distortion and rotation of the volume element.

case(ii):when curl $u = 0$, the identity

curl curl $u =$ grad div $u - \nabla^2 u$

$\Rightarrow$ grad div u $= \nabla^2 u$

The general equation becomes

$(\lambda + 2\mu)v^2 u = \rho u_{tt}$

$$u_{tt} = \frac{\lambda + 2\mu}{\rho}\nabla^2 u$$

$$\Rightarrow u_{tt} = e^2\nabla^2 u \quad \dots\dots\dots\dots \ (16)$$

where velocity of propagation is $c = \sqrt{\frac{\lambda+2\mu}{\rho}}$

This is the care of an irrational or dilatational wave equation propagation, $\because$ curl $u = 0$ describes irrotational motion.

Equation (15) & (16) are called the three -dimensional wave equation.

1.5. conduction of heat in Solids:

Derivation of Heat Equation:

Let a domain D^* be bounded by a closed Surface B^*. Let D be an arbitrary bounded by closed surface B in D^*

Let $U(x, y, z, t)$ be the temperature at a point heat flows from places of higher temperature to places of lower temperature.

Now by Fourier's law, ' v ' which is the heat of flow of heat (or) velocity of heat flow is proportional to gradient of temperature

(i.e.) $v\alpha$ grad u (or) $v = -k$ grad u ... . (1)

Where constant k is called the thermal conductivity of the body.

[v=rate of heat flow=velocity of heat flow=$-k$ grad u]…..(1)

Now, amount of heat leaving D per unit time = rate of decrease of heat in D

Now, Amount of heat leaving D per unit time

$$= \iint_B (v \cdot n)ds \quad = \iiint_D dw(v)\, dx\, dy\, dz$$

$$= \iiint_D (\nabla \cdot v)dxdydz \quad \iiint_D (\nabla \cdot (-kgradu))dxdydz \text{ by}(1)$$

2

$$= -k \iiint_D \nabla^2 u\, dx\, dy\, dz$$

$\therefore$ Amount of heat leaving D per lent time

=-k$\iiint_D \nabla^2 u\, dx\, dy\, dz$ …………(2)

Now, amount of heat in $D = \iiint \sigma\rho u dxdydz$

Where ρ is the density of the material of the body & σ is its specific hook.

$\therefore$ rate of decrees of heat in $D = -\frac{\partial}{\partial t}\left(\iiint \sigma\rho u dxdydz\right)$

$$= -\iint_D \int \sigma\ \rho \frac{du}{\partial t} dxdydz \quad \ldots\ldots\ldots (3)$$

using (2) & (3) we get $-k \iiint_D \nabla^2 u dxdydz = -\iiint_D \sigma\rho \frac{\partial u}{\partial t} dxdydz$

$\iiint_D [\sigma\rho u_t - k\nabla^2 u] dydxdz = 0$ ……….......(4)

Assuming integrand in (4) is continuous.

(4) gives

The integrand $\sigma\rho ut - k\Delta^2 u = 0$

(or) $u_t = k\nabla^2 u$ ……….(5) when $k = \frac{k}{\sigma\rho}$

1.6. The Gravitational Potential:

Laplace equation:

- Laplace equation (or) Potential equation
- Laplace equation is a 2^{nd} order partial differential eq.
- Named after French scholar P.S. Laplace
- Laplace equation appears in many physical problems such as in problem of electrical, magnetic & gravitational potential etc.
- Laplace equation in three dimensions is $\nabla^2 u = 0$, $\nabla \cdot \nabla u = 0$ (or) div grad $u = 0$

where ∇ is the divergence operator (or div) ∇ is the gradient operator (or grad) $u(x, y, z)$ is a twice differentiable real valued function

- Del operation $\nabla \rightarrow \nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} i + \frac{\partial}{\partial z} k$
- Laplace operator $\nabla^2 = \nabla \cdot \nabla$

$$\nabla^2 = \nabla \cdot \nabla$$
$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)$$
$$\left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)$$
$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- For a function $u(x, y, z)$ grad $u = \nabla u = \frac{\partial u}{\partial x} i + \frac{\partial u}{\partial y} j + \frac{\partial u}{\partial z} k$ is a vector.

div grad $u = \nabla \cdot \nabla u$ is a scalar quantity. So in Laplace equation div grad $u = 0$

- Between two objects (or) particle in the universe there is a force that tends to attract them towards each other.
- This force is gravity or gravitational force
- Every object in the universe is surrounded by a gravitational field.
- Newton's low of Gravitation

  The gravitational force $F = \frac{GmM}{r^2}$ $(= F_1 = F_2)$ acts along the line joining their centers. mass m & M at a distance reappear G is the gravitational constants
- Gravitational potential:

Gravitational potential (or) potential at a point Q due to a particle of man m at a point P It's the workdone in bringing a particle of unit mass form infinity up to the point Q, by the force M of particle of mass m

$m = \frac{-Gm}{r}$ where $r = pQ$

Gravitational potential at a point Q due to a system of particle of masses $m_1, m_2, \ldots m_n$ at points $P_1, P_2, \ldots P_n$

Its the work done in bringing a particle of unit mass from infinity upto the point Q, by the force of attraction of masses $m_1, m_2, \ldots, m_n$

$mV = -\mathrm{G} \sum_{i=1}^{n} \frac{m_i}{r_i}$,where, $r_1 = P_1Q$, $r_2 = P_2Q \ldots\ldots$ $r_n = P_nQ$

Derivation of Laplace's Equation for Gravitational:

Potential:

consider a particle of mass m placed at a Point $P(a, b, c)$ at a distance r from point $Q(x, y, z)$.

The potential V, at a point Q due to mars ' m ' is $V = \frac{-Gm}{r}$

Now,

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 \ldots\ldots\ldots (1)$$

$$2r\frac{\partial r}{\partial x} = 2(x-a)$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{2(x-a)}{2r} = \frac{x-a}{r} \ldots\ldots\ldots\ldots (2)$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y-b}{r} \ldots\ldots\ldots (3)$$

$$\frac{\partial r}{\partial z} = \frac{z-c}{r} \text{...........}(4)$$

Now, as $V = \frac{-Gm}{r}$

$$\Rightarrow \frac{\partial v}{rx} = -\text{Gm}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x}$$

$$\frac{\partial v}{\partial x} = \frac{Gm}{r^2}\frac{(\text{x}-\text{a})}{\text{r}} \qquad \text{...............} (5)$$

Similarly,

$$\frac{\partial v}{\partial y} = \frac{Gm}{r^3}(y-b) \dots\dots\dots(6) \qquad \frac{\partial V}{\partial z} = \frac{Gm}{r^3}(z-c) \dots\dots\dots\dots(7)$$

Now,

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{Gm}{r^3}(x-a)\right]$$

$$= Gm\left[\left(\frac{-3}{r^4}\frac{\partial r}{\partial x}(x-a)\right) + \frac{1}{r^3}\right]$$

$$= \text{Gm}\left[\frac{1}{r^3} - \frac{3}{r^4}\frac{(x-a)}{r} \cdot (x-a)\right]$$

$$\frac{\partial^2 v}{\partial x^2} = \text{Gm}\left[\frac{1}{r^3} - \frac{3}{r^5}(x-a)^2\right] \qquad \text{................} (8)$$

Similarly, $\frac{\partial^2 v}{\partial y^2} = \text{Gm}\left[\frac{1}{r^3} - \frac{3}{r^5}(y-b)^2\right]$ ....(9)

$$\frac{\partial^2 v}{\partial z^2} = \text{Gm}\left[\frac{1}{r^3} - \frac{3}{r^5}(z-c)^2\right] \qquad \text{................} (10)$$

Adding (8),(9),(10) we

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{3Gm}{r^3} - \frac{3Gm}{r^5}[(x-a)^2 + (y-b)^2$$

$$= \frac{3Gm}{r^3} - \frac{3Gm}{r^5}r^2$$

$$= \frac{3Gm}{r^3} - \frac{3Gm}{r^3}$$

$$\text{(or) } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)v = 0$$

$\Rightarrow \nabla^2 \text{V} = 0$ which is Laplace's equation.

Note: When there are particles of mess ' m; ' placed at a points $p(a,b,c)$ at a distance r_i from point $Q(x,y,z)$.

$$\frac{\partial r_i}{\partial x} = \frac{(x-a_i)}{r_i}, \frac{\partial r_i}{\partial y} = \frac{(y-b_i)}{r_i}, \frac{\partial r_i}{\partial z} = \frac{(z-c_i)}{r_i},$$
$$i = 1,2,3 \ldots n$$

As $V = -G\sum_{i=1}^{n} \frac{m_i}{r_i}$

$$\frac{\partial v}{\partial x} = G\sum_{i=1}^{n} \frac{m_i}{r^3}(x-a_i)$$
$$\frac{\partial v}{\partial y} = G\sum_{i=1}^{n} \frac{m_i}{r_i{}^3}(y-b_i)$$
$$\frac{\partial v}{\partial z} = G\sum_{i=1}^{n} \frac{m_i}{\partial_i{}^3}(z-c_i)$$
$$\frac{\partial^2 v}{\partial x^2} = G\sum_{i=1}^{n} \left[\frac{m_i}{r_i^3} - \frac{3m_i}{r_i 5}(x-a_i)^2\right]$$
$$\frac{\partial^2 V}{\partial y^2} = G\sum_{i=1}^{n} \left[\frac{m_i}{r_i^3} - \frac{3m_i}{r_i 5}(y-b_i)^2\right]$$
$$\frac{\partial^2 v}{\partial z^2} = G\sum_{i=1}^{n} \left[\frac{m_i}{r^3} - \frac{3m_i}{r_i^5}(z-c_i)^2\right]$$
$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = G\sum_{i=1}^{n} \left[\frac{3m_i}{r_i^3} - \frac{3m_i}{r_i 5}\right.$$
$$\left.[(x-ai)^2 + (y-bi)^2 + (z-c_i)^2]\right]$$
$$= G\sum_{i=1}^{n} \left[\frac{3m_i}{r_i} - \frac{3m_i}{r_i} r_i{}^2\right]$$
$$= G\sum_{i=1}^{n} \left[\frac{3m_i}{r_i^3} - \frac{3m_i}{r_i^3}\right], V = 0$$
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)V$$
$$\nabla^2 V = 0$$

Which is the laplace equation.

Exercises Problem:

Various forms of wave equation

1.Show that the equation of motion of a long string $= c^2u_{xx} - g$ Where g is the gravitational accelation.

Solution:

consider a small element ab of the given string. To find the eq. of motion, the following assumptions are made

1. The tension in the string is always in the direction of the tangent to the existing profile of the string
2. The tension is constant
3. The weight of the string is small compared with the tension in the string
4. The deflection is small compared with the length of the string.
5. The slope of the displaced string at any point is small compared with unity.
6. There is only pure transverse vibration.

As acceleration due to gravity, (i.e.) g acts on the string. Therefore, the resultant force acting on the element ab of the string in the vertical direction is

$T\sin\beta - T\sin\alpha - \rho\Delta s g \rightarrow (1)$

As slope of displaced string is small

$\Delta s \simeq \Delta r$

And by Newton's 2nd law of motion

$$T\cdot\sin\beta - T\sin\alpha - \rho\Delta s g = \rho\Delta s u_{tt} \dots\dots\dots (2)$$

Since the angle α, β are small

$\sin\alpha \simeq \tan\alpha, \sin\beta \simeq \tan\beta$

$\therefore (2) \Rightarrow T\tan\beta - T\tan\alpha - \rho\Delta x g = \rho\Delta x utt$

$$\tan\alpha = U_x(x,t)$$
$$\tan\beta = U_x(x+\Delta x, t)$$

$\tau u_x(x,t) - \tau u_x(x+\Delta x, t) - \rho\Delta x g = \rho\Delta x u_{tt}$

$$\frac{T}{\rho\Delta x}[u_x(x,t) - u_x(x+\Delta x,t)] - g = u_{tt}$$

$$\frac{T}{\rho}\left[\frac{u_x(x,t') - u_x(x+\Delta x,t)}{\Delta x}\right] - g = u_{tt}$$

taking $\Delta x \to 0$

$$\frac{T}{p}u_{xx} - g = U_{tt}$$

$\therefore u_{tt} = c^2 u_{xx} - 9$ when $c^2 = \frac{T}{\rho}$ ………. (3)

$\therefore$ equation (3) is the form of wave equation for a a vibrating string, when gravitational acceleration is taken in to consideration.

2. Derive the damped wave equation of a string u_{tt} +a$u_t = c^2 U_{xx}$ where the damping force is proportional to the velocity & a is constant. Considering a restoring force is proportional to the displacement of a string Show that the resulting en is $u_{tt} + au_t + bu = c^2 u_{xx}$ where b is a constant. This equation is called telegraph equation.

Solution:

Damped wave equation:

To discuss problem of vibrating string, when a damping force acts on the string.

Damping force:

1. A damping force is any force, such as air resistance which acts when a child is swinging, which prevents the vibratory motion of a body. In the care of a vibrating string, a damping force tries to bring the string in rest position by dissipation of energy.
2. The damping force is directly proportion d to the velocity u_t and the magnitude of the damping force F_1 at time t is g_n by Damping coefficient time velocity at time t (i.e.) $F_1 = -au_t \to (1)$

Where a negative sign is there as the direction of the damping force is opposite to the direction of velocity at time t.

The resultant force acting on the element ab of the string in the vertical direction is

$$T\sin\beta - T\sin\alpha + \rho\Delta s F_1$$
$$\text{(i.e.) } T\sin\beta - T\sin\alpha + \rho\Delta S(-au_t)$$

$\because$ The angles α, β are small

$$\sin\alpha \simeq \tan\alpha, \sin\beta \simeq \tan\beta, \Delta s = \Delta x$$

$\therefore T\tan\beta - T\tan\alpha - \rho\Delta x \text{ a}u_t$ … … … … (2)

By Newton's 2nd law of motion

$$T\tan\beta - T\tan\alpha - \rho\Delta x a u_t = \rho\Delta x u_{tt}$$

$$\Rightarrow \frac{T}{\rho\Delta x}[\tan\beta - \tan\alpha] - au_t = u_{tt}$$

$$\Rightarrow \frac{T}{\rho}\left[\frac{\tan\beta - \tan\alpha}{\Delta x}\right] - au_t = u_t t\Big) \;\ldots\ldots\ldots.(3)$$

$\tan\alpha = u_x(x,t); \tan\beta = u_x(x+\Delta x, t)$

$$\therefore \frac{T}{\rho}\left[\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}\right] - au_t = u_{tt}$$

Taking limit $\Delta x \to 0$

$$\frac{T}{\rho}u_{xx} - au_t = u_{tt}$$

$\therefore u_{tt} = -au_t + c^2 u_{xx}$ where $c^2 = \frac{T}{\rho}$

$$u_{tt} = c^2 u_{xx} - au_t \;\ldots\ldots\ldots..(4)$$

(4) is the Dumped wave equation.

Telegraph Equation:

To discuss problem of vibrating string When a damping force as well as a restoring force acts on the string.

Restoring force:

1. A restoring force is any force such as gravity acting on the pendulum when it swings which acts to bring a body to its equilibrium position
2. The restoring force is a function of position ' u ' only k it is always directed back towards the equilibrium position of the system.
3. The restoring force F_2 is given by $F_2 = -bu$ $\ldots\ldots\ldots\ldots..(5)$

When both the Damping force F_1 as well as the restoring force F_2 act on the string then:

The resultant force acting on the element ab of thestring in the vertical direction is

$T\sin\beta - T\sin\alpha + \{p\Delta s(F_1 + F_2)\}$

By newton's 2nd law of motion

$T\sin\beta - T\sin\alpha + \rho\Delta(F_1 + F_2) = \rho\Delta s u_{tt}$ ………….(7)

As slope of displaced string is small

$\Delta s \simeq \Delta x$

$\because$ The angles α & β are small

$\sin\alpha \simeq \tan\alpha \; \sin\beta \simeq \tan\beta$

$\therefore (7) \Rightarrow T\tan\beta - T\tan\alpha + \rho\Delta x(F_1 + F_2) = \rho\Delta x u_{tt}$

$\tan\alpha = u_x(x,t)$

$\tan\beta = u_x(x+\Delta x,t)$

$\therefore Tu_x(x+\Delta x,t) - Tu_x(x,t) + \rho\Delta x(-aut - bu)$

$= \rho\Delta x u_{tt}$

$\Rightarrow \frac{T}{\rho\Delta x}[u_x(x+\Delta x,t) - u_x(x,t)] - \frac{\rho\Delta x}{\rho\Delta x}(au_t + bu)$

$= u_{tt}$

$$\Rightarrow \frac{T}{\rho}\left[\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}\right] - au_t - bu$$

$$= u_{tt}$$

taking limit $\Delta x \to 0$

$$\frac{T}{\rho}u_{xx} - au_t - bu = u_{tt}$$

$\Rightarrow u_{tt} = c^2u_{xx} - au_t - bu \quad \dots\dots\dots.(8)$

where $c^2 - T/\rho$

$\Rightarrow$ (8) is also called telegraph eqn.

Classification of Second Order Equations:

1.7. Second - order equation in two independent variables:

consider the PDE of 2nd order in two independent variables x & (dependent variable u) as

$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = G \quad \dots\dots\dots(1)$

Where the coefficients are functions of x & y and do not vanish simultaneously & the function u & A, B, C, D, E,F,G are twice continuously differentiable in some domain R.

The classification of 2nd order equations is based upon the possibility of reducing (1) by a coordinate transformation to canonical (or) standard form at a point.

An equation is said to be hyperbolic, parabolic, or elliptic at a point (x_0, y_0) according as

$B^2 - 4AC > 0, = 0, < 0$, at $(x_0, y_0) \quad \dots\dots\dots\dots..(2)$

If it is true at all points, then the equation, is said to be hyperbolic, parabolic, or elliptic in a domain.

To transform (2) to a canonical form we make a change of independent variables. Let the new variable be $\xi = \xi(x,y)$, $\eta = h(x,y)$ ………..(3)

Assuming that ξ & η are twice continuously differentiable and that the Jacobian

$J = \begin{vmatrix} \xi x & \xi y \\ \eta x & \eta y \end{vmatrix}$ ……….. (4) is non zero in the region under consideration, then xky can be determined uniquely from the system (3) Let $x \times y$ be twice continuously differentiable function of ξ & n.

Then we have

$$\left.\begin{aligned} u_x &= u_\xi \xi x + u_\eta \eta_x \\ u_y &= u_\xi \xi y + u_\eta \eta_y \\ U_{xx} &= U_{\xi\xi}\xi x^2 + 2U\xi\eta\xi_x\,\eta_x + U_{\eta\eta}\eta_x^2 \\ &\quad + U_\xi \xi_{xx} + U_\eta \eta_{xx} \\ U_{xy} &= U_{\xi\xi}\xi_x\xi_y + U_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) \\ &\quad + U_{\eta\eta}\eta_x\eta_y + U_{\xi\xi xy} + U_\eta \eta_{xy} \\ U_{yy} &= U_{\xi\xi}\xi_y^2 + 2U_{\xi\eta}\xi_y\eta_y + U_{\eta\eta}\eta_y{}^2 + U_\xi \xi_{yy} + u_\eta \eta_{yy} \end{aligned}\right\} \text{……… (5)}$$

Substituting (5) in (1) we obtain

$$A^* U_{\xi\xi} + B^* U_{\xi\eta} + C^* U_{\eta\eta} + D^* U_\xi + E^* U_\eta + F^* U = G^* \quad \text{………… (6)}$$

where

$$\begin{aligned} A^* &= A\xi\, x^2 + E\xi x\xi y + c\xi^2 y \\ B^* &= 2A\,\xi_x\,\eta_x + B(\,\xi_x\,\eta_y + \,\xi_y\,\eta_x) + \,\,2C\,\xi_y\,\eta_y \\ \\ c^* &= A\eta x^2 + B\eta x n y + c\eta y^2 \\ D^* &= A\xi xx + B\xi xy + C\xi yy + D\xi x + E\xi y \\ E^* &= A\eta_{xx} + Bn_{xy} + C\eta_{yy} + D\,\eta_x + E\,\eta_y \\ F^* &= F \\ G^* &= G \end{aligned}$$

The classification of (1) depends on the coefficients A, B, C, at a given point (x, y).

we shall rewrite (1) as

$$AU_{xx} + BU_{xy} + CU_{yy} = H \quad \text{… … … … … (8)}$$

where $H = H(x, y, u, u_x, u_y)$

equation (6) as $A^* U_{\xi\xi} + B^* U_{\xi\eta} + C^* U_{2n} = 4^*$ … … … … .. (9) where $n^* = H^*(\xi, n, u, u_\xi, u_n)$

1.8. Canonical form

Transforming first order linear PDE into Canonical form.

Step 1:

compare the given PDE with the standard PDE

$$AU_{xx} + BU_{yy} + CU_{yy} + DU_x + FU_y + FU = G \text{ … … …. (1)}$$

Step 2:

Find the discriminant $B^2 - 4AC$ and classify the given PDE as follows.

Discriminant	PDE
$B^2 - 4AC > 0$	hyperbolic
$B^2 - 4AC = 0$	parabolic
$B^2 - 4AC < 0$	elliptic

Step 3:

Find the characteristic equations:

PDE Characteristic equation

Hyperbolic

$$\left.\begin{aligned}\frac{dy}{dx} = \frac{-\xi x}{\xi y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \\ \frac{dy}{dx} = \frac{-\eta x}{2y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}\end{aligned}\right\} \ldots\ldots\ldots\ldots (2)$$

Parabolic $\frac{dy}{dx} = \frac{-\xi tx}{\xi y} = \frac{B}{2A} \ldots\ldots\ldots\ldots (3)$

Elliptic
$$\begin{aligned}\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} \\ \frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A}\end{aligned} \quad \ldots\ldots\ldots\ldots(4)$$

step 4:

Integrate the characteristic equations to obtain, $\begin{aligned}\xi(x, y) = \varphi_1(x, y) = c_1 \\ \eta(x, y) = \varphi_2(x, y) = c_2\end{aligned}$

hyperbolic $\xi(x, y) = c_1$ & $\eta(x, y) = c_2$

parabolic $\xi(x, y) = c$, & η is chosen such that it is not parallel to the ξ-coordinate

(i.e.) η is chosen such that the Jacobian of the transformation is not zero

Elliptic $\xi(x, y) = c_1$& $\eta(x, y) = c_2$

Introduce the 2nd trans formation such that $\alpha = \frac{\xi + n}{2}$ & $\beta = \frac{\xi - \eta}{2i}$

Step 6:

The transformed canonical equation n is given by

$\Rightarrow$ (2) $A^* U_{\xi\xi} + B^* U_{\xi\eta} + C^k u_{\eta\eta} + D^* U_{\xi} + E^* U_{\eta} + F^* U = G^*$

Where,

$A^* = A\xi\ x^2 + E\xi x\xi y + c\xi^2 y$
$B^* = 2A\ \xi_x\ \eta_x + B(\ \xi_x\ \eta_y +\ \xi_y\ \eta_x) +\ \ 2C\ \xi_y\ \eta_y$

$c^* = A\eta x^2 + B\eta xny + c\eta y^2$
$D^* = A\xi xx + B\xi xy + C\xi yy + D\xi x + E\xi y$
$E^* = A\eta_{xx} + Bn_{xy} + C\eta_{yy} + D\ \eta_x + E\ \eta_y$
$F^* = F$
$G^* = G$

Equation (2), (3) & (4) which are known as the characteristic equations, are the ODE for families of curves in the xy-plane along $\xi =$ constant &$\eta =$ constant. The integral of (5) are called the characteristic curves.

Since the equations are 1st order ODE, the solution may be written as $\varphi_1(x, y) = c_1 \& \varphi_2(x, y) = c_2$.

Hence the transformation $\xi = \varphi_1(x, y) < \eta = \varphi_2(x, y)$ will transform

$AU_{xx} + BU_{xy} + CU_{yy} = H.$

where $H = H(x, y, u, u_x, u_y)$ to canonical form.

A. Hyperbolic Type:

If $B^2 - 4AC > 0$, then integration of $\frac{dy}{dx} = \frac{B+\sqrt{B^2-4AC}}{2A} * \frac{dy}{dx} = \frac{B-\sqrt{B^2-4AC}}{2A}$ yields two real k distinct families of characteristics.

Equation, $A^*U_{\xi\xi} + B^*U_{\xi\eta} + c^*U_{\eta\eta} = A^*$
where $x^* = x^*(x, y, u, u_\xi, u_\eta)$

reduces to $U_{\xi\eta} = n_1$ where $H_1 = H^*/B^*$.

It can be easily shown that $B^* \neq 0$. This form is called the first canonical form of the hyperbolic equation.

Now if the new independent variable $\alpha = \xi + \eta$ & $\beta = \xi - \eta$ are introduced then $U_\xi\eta = n_1$ is transformed into $U_{\alpha\alpha} - U_{\beta\beta} = n_2(\alpha, \beta, U, u_\alpha, U_\beta)$

This form is called the 2nd canonical form of the hyperbolic equation.

B. Parabolic Type:

For the parabolic equation the discriminant

$B^2 - 4AC = 0$ which can be if $B^* = 0$ & A^* or $C^* = 0$

suppose $A^* = 0$ then $A^* = A\xi x^2 + B\xi x2y + C\xi y^2 = 0$

$\Rightarrow A\xi x^2 + B\xi x\xi y + C\xi y^2 = 0$

$$\div by \xi y^2 \; A\left(\frac{\xi x}{\xi y}\right)^2 + B\left(\frac{\xi x}{\xi y}\right) + C = 0$$

$$\therefore \frac{\xi x}{\xi y} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-B}{2A} \; (\because B^2 - 4AC = 0)$$

Hence we find the equation $\xi(x, y) = \varphi_1 = c_1$

$B^2 - 4AC$ we get $\frac{dy}{dx} = \frac{B}{2A} \times$ get the implicit solution $\xi(x, y) = c_1, \eta = y$ or any other function independent of x We can verify that $A^* = 0, B^* = 0$ as follows:

$$B^* = 2A\xi x\, \eta_x + B(\xi x\, \eta_y + \xi y\, \eta_x) + 2C\xi y\, \eta_y$$
$$= 2A\xi x\, \eta_x + 2\sqrt{A}\sqrt{C}\xi x\, \eta_y + 2\sqrt{A}\sqrt{C}\xi y\, \eta_x$$
$$+2c\xi y\, \eta_y$$

$$= 2\sqrt{A}\xi x(\sqrt{A}\, \eta_x + \sqrt{c}\, \eta_y) + 2\sqrt{c}\xi y(\sqrt{A}\eta x + \sqrt{c}\eta y)$$

$$= (\sqrt{A}\eta x + \sqrt{c}\eta y)\left(2\sqrt{A}\xi x + 2\sqrt{c}\eta_y\right)$$
$$= 2(\sqrt{A}\eta x + \sqrt{c}\eta y)(\sqrt{A}\xi x + \sqrt{c}\xi y)$$
$$= 2[-\sqrt{c}\xi y + \sqrt{c}\xi y][\sqrt{A}\eta x + \sqrt{c}\eta_y]$$

$$= 0 \left[\because \frac{\xi x}{\xi y} = \frac{-B}{2A} = \frac{-2\sqrt{A}\sqrt{c}}{2A}\right.$$
$$= \frac{-\sqrt{c}}{A}$$
$$(\sqrt{A}\xi x = -\sqrt{c}\xi y)$$

$\therefore A^* = B^* = 0$ then η can be chooses in anyway

we take as long as it is not parallel to ξ coordinate

In other words, we choose η there Jacobian of the transformation is not equal to zeno.

Thus we can write the canonical equation for parabolic care by substituting & $> h$ in

by substiting,

$$AU_{\xi\xi} + B \cup \xi\eta + CU_{\eta\eta} + DU\xi + EU_\eta + F = 0$$

which reduce to either $U_{\xi\xi} = H_1\left(\xi, \eta, U, U_\xi, U_n\right)$

(on) $U_{\eta\eta} = H_2(\xi, \eta, v, U\xi, U\eta)$.

This is called the canonical form of the parabolic equation.

C. Elliptic type:

For the elliptic case $B^2 - 4AC < 0$, The characteristic equation $\frac{dy}{dx} = \frac{B-\sqrt{B^2-4AC}}{2A}, \frac{dy}{dx} = \frac{B+\sqrt{B^2-4AC}}{2A}$ gives us complex conjugate coordinate ξ, η.

Now we to make another transformation from (ξ, η) to (α, β) where $\alpha = \frac{\xi+h}{2}, \beta = \frac{\xi-h}{2i}$ which gives the required canonical equation in the form of elliptic equation

$U_{\alpha\alpha} + U_{\beta\beta} = H(\alpha, \beta, U, U_\alpha, U_\beta)$

Example 1:

Reduce the equation $y^2 u_{xx} - x^2 u_{yy} = 0$ into canonical form (or) Determine the region in which the equation. $y^2 u_{xx} - x^2 U_{yy} = 0$ is hyperbolic, parabolic, elliptic, and trans form the equation in the respective region to the canonical form.

Solution:

consider the equation $y^2 \cup_{xx} - x^2 v_{yy} = 0$ …………(1) compare this in to the general equation.

$AU_x x + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = G$ ………… (2)

Step 1:

Here $A = y^2, B = 0, C = -x^2, D = E = F = G = 0$

$$\begin{aligned} B^2 - HAC &= 0 + 4x^2y^2 \\ &= 4x^2y^2 \\ &= (2xy)^2 > 0. \end{aligned}$$

∴ The equation is hyperbolic everywhere except on the coordinate axes $x = 0$ & $y = 0$

Step 2 :

The characteristic equations is $\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$

$$= \frac{\pm\sqrt{4x^2y^2}}{2y^2} = \frac{\pm 2xy}{2y^2}$$

$$= \pm\frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y} \ \& \ \frac{dy}{dx} = \frac{-x}{y}$$

$$\therefore ydy = xdx \Rightarrow \& ydy + xdx = 0$$

Step :3

integrating, we get $\frac{y^2}{2} = \frac{x^2}{2} + c_1$ …………(3)

$\frac{y^2}{2} + \frac{x^2}{2} = c_2$ ………… (4)

equation (3) is a family of hyperbolas

equation (4) is a family of circles.

step :4

To t transform the given equation to the canonical form

Let $\xi = \frac{1}{2}y^2 - \frac{1}{2}x^2$

$$\eta = \frac{1}{2}y^2 + \frac{1}{2}x^2$$

$$\xi x = \frac{-2x}{2} = -x \quad \xi xx = -1 \; \xi xy = 0$$

$$\xi y = \frac{2y}{2} = y \quad \xi yx = 0 \quad \xi yy = 1$$

$$\eta_x = \frac{2x}{2} = x \quad \eta_{xx} = 1 \; \eta_{xy} = 0$$

$$\eta_y = \frac{2y}{2} = y \; \eta_{yx} = 0 \; \eta_{yy} = 1$$

step: 5

$$U_x = U_\xi \xi_x + U_\eta \eta_x$$
$$= -xU_\xi + xU_n$$

$$U_y = U_\xi \xi_y + u_\eta \eta_y$$
$$= yU_\xi + yu_n$$

$$U_{xx} = U_{\xi\xi}\xi x^2 + 2U_{\xi y}\xi x\eta_x + u_{\eta\eta}\eta_{y2}$$
$$+u_\xi \xi xx + u_\eta \eta_{xx}$$
$$= (-x)^2 u_{\xi\xi} + 2(-x)(x)u_{\xi y} + u_{2\eta}(x)^2$$
$$+u_\xi(-1) + u_\eta(1)$$
$$= x^2 u_{\xi\xi} - 2x^2 u_{\xi y} + x^2 u_{\eta\eta} - u_\xi + u_\eta$$

$$U_{xy} = U_{\xi\xi}\xi_y + U_{\xi\eta}(\xi x\eta_y + \xi y\eta_x)$$
$$+U_{\eta\eta}\eta_x\eta_y + U_\xi \xi xy + U_\eta \eta_{xy}$$
$$= U_{\xi\xi}(-x)(y) + U_{\xi\eta}((-x)(y) + (y)(x))$$
$$+U_{\eta\eta}(x)(y) + U_\xi(0) + U_\eta(0)$$
$$= -xyU_{\xi\xi} + U_{\eta\eta}xy$$

$$U_{yy} = U_{\xi\xi}\xi y^2 + 2U_{\xi\eta}\xi y\eta y + U_{n\eta}\eta_y^2 + U_\xi \xi yy + U_n\eta_y$$
$$= U_{\xi\xi}(y)^2 + 2U_{\xi\eta}(y)(y) + U_{n\eta}(y)^2 + U_\xi(1) + U_n(1)$$
$$= y^2 U_{\xi\xi} + 2y^2 U_{\xi\eta} + y^2 U_{\eta\eta} + U_\xi + U_\eta$$

Step:6

Sub. in (2) we get

$$A\left(x^2 \cup_{\xi\xi} - 2x^2 U_{\xi\eta} + x^2 U_{n\eta} - U_\xi + U_n\right)$$
$$+B(-xyU_{\xi\xi} + xy\, U_{n\eta}) + C\left(y^2 U_{\xi\xi} + 2y^2 u_{\xi\eta}\right.$$
$$\left.+y^2 u_{\eta\eta} + u_\xi + u_\eta\right)$$
$$+D\left(-x \cup_\xi + x \cup_\eta\right) + E\left(y \cup_\xi + y \cup_\eta\right) + F = 0$$
$$y^2\left(x^2 U_{\xi\xi} - 2x^2 U_{\xi\eta} + x^2 \cup \eta\eta - u_\eta + U_n\right) + 0$$
$$-x^2\left(y^2 \cup u_\xi + 2y^2 U_{\xi\eta} + y^2 u_{2\eta} + u_\xi + u_\eta\right) = 0$$
$$x^2 y^2 v_\xi \xi - 2y^2 x^2 \cup \xi\eta + x^2 y^2 v_{n\eta} - y^2 u_\xi + y^2 u_n$$
$$-x^2 y^2 U_{\xi\xi} - 2y^2 x^2 \cup \xi\eta - x^2 y^2 \eta_{\eta n} - x^2 \cup \xi$$
$$-x^2 \cup \eta = 0$$
$$-4x^2 y^2 u_\xi \eta - u_\xi (x^2 + y^2) + u_\eta (y^2 - x^2) = 0$$
$$u_{\xi\eta} + U_\xi \frac{x^2 + y^2}{4x^2 y^2} - \frac{u_\eta (y^2 - x^2)}{4x^2 y^2} = 0$$
$$U_{\xi\eta} = -U_\xi \frac{x^2 + y^2}{4x^2 y^2} + \frac{U_\eta (y^2 - x^2)}{4x^2 y^2}$$
$$= -U_\xi \frac{2n}{-\mu(\xi^2 - \eta^2)} + U_\eta \left(\frac{2\xi}{-4(\xi^2 - \eta^2)}\right)$$
$$U_{\xi\eta} = U_\xi \frac{n}{2(\xi^2 - \eta^2)} - U_\eta \frac{\xi}{2(\xi^2 - \eta^2)}$$

which is the required canonical

form of the hyperbolic equation.

Example 2:

Reduce the equation $x^2 U_{xx} + 2xyU_{xy} + y^2 U_{yy} = 0$ (or) Determine the region in which the equation $x^2 Uxx + 2xy \cup xy + y^2 \cup yy = 0$ is hyperbolic, parabolic, elliptic, x trans form the equation into canonical form.

Solution:

consider the equation $x^2 U_{xx} + 2xy \cup xy + y^2 U_{yy} = 0 \dots\dots\dots\dots (1)$ compar this equation into the general PDE $AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = G \dots\dots\dots\dots$ (2)

Step 1:

$$A = x^2\,,\ B = 2xy\,,\ C = y^2\,,\ \ D = E = F = G = 0$$
$$B^2 - 4AC = (2xy)^2 - 4x^2 y^2 = -2x^2 y^2 + 4x^2 y^2 = 0$$

The equation is parabolic everywhere.

Stop -2:

The characteristic equation is $\frac{dy}{dx} = \frac{B \pm \sqrt{B^2-4AC}}{2A}$

$$= \frac{2xy \pm \sqrt{(2xy)^2 - 4(x^2)y^2)}}{2(x^2)}$$

$$\frac{dy}{dx} = \frac{2xy}{2x^2} = \frac{y}{x}$$

$$xdy - ydx = 0$$

$$d\left(\frac{y}{x}\right) = 0$$

step: 3

integrating, we get $\frac{y}{x} = c_1 \rightarrow (3)$

Which is the family of straight line

step :4

To transform the given equation to the canonical form. Let $\xi = \frac{y}{x}$ & $\eta = y$, where η is arbitrary.

$$J = \begin{vmatrix} \xi x & \xi y \\ hx & 2y \end{vmatrix} = \begin{vmatrix} \frac{-y}{x^2} & \frac{1}{x} \\ 0 & 1 \end{vmatrix} = \frac{-y}{x^2} \neq 0$$

$$\xi_x = \frac{-y}{x^2} \quad \xi xx = \frac{+2y}{x^3} \quad \xi xy = \frac{-1}{x^2}$$

$$\xi_y = \frac{1}{x} \quad \xi yx = \frac{-1}{x^2} \quad \xi yy = 0$$

$$\eta_x = 0 \quad \eta_{xx} = 0 \quad hxy = 0$$

$$\eta_y = 1 \quad \eta_{yx} = 0 \quad h_{yy} = 0$$

Step 5:

$$U_x = U_\xi \xi_x + U_\eta \eta_x$$

$$= U_\xi \left(\frac{-y}{x^2}\right) + U_n(0)$$

$$= \frac{-y}{x^2} U_\xi$$

$$U_y = U_\xi \xi_y + U_n \eta_y$$

$$= U_\xi \left(\frac{1}{x}\right) + U_n(1)$$

$$= \frac{1}{x} U_\xi + U_n$$

$$U_{xx} = U_{\xi\xi} \xi x^2 + 2U_\eta \xi x \eta_x + \mathrm{U}\eta\eta x^2 + U_\xi \xi xx + U_\eta \eta xx$$

$$= \frac{y^2}{x^4} u_{\xi\xi} + 2u_{\xi\eta}\left(\frac{-y}{x^2}\right)(0) + u_{\eta\eta}(0)^2 + u_\xi\left(\frac{+2y}{x^3}\right) + u_\eta(0)$$

$$= \frac{y^2}{x^4} u_{\xi\xi} + \frac{2y}{x^3} U_\xi$$

$$U_{xy} = U_{\xi\xi} \xi x \xi y + U_{\xi\eta}(\xi \eta_y + \xi_y \eta_x) + U_{\eta\eta} \eta_x \eta_y$$
$$+ U_\xi \xi xy + u_\eta \eta xy$$

$$= \left(\frac{-y}{x^2}\right) \cdot \frac{1}{x} U_{\xi\xi} + U_{\xi\eta}\left(\frac{-y}{x^2}(1) + \left(\frac{1}{x}\right)(0)\right) + U_{\eta\eta}(0)(1)$$
$$+ u_\xi \left(\frac{-1}{x^2}\right) + u_\eta(0)$$

$$= \frac{-y}{x^3} U_{\xi\xi} - \frac{y}{x^2} U_{\xi\eta} - \frac{1}{x^2} U_\xi$$

$$U_{yy} = U_{\xi\xi} \xi y^2 + 2U_{\xi\eta} \xi y \eta y + U_{\eta\eta} \eta y^2 + U_\xi \xi yy$$
$$+ u_\eta \eta_{yy}$$

$$= \frac{1}{x^2} U_{\xi\xi} + 2\frac{1}{x} U_{\xi\eta} + U_{\eta\eta} + U_\xi(0) + 0$$

$$= \frac{1}{x^2} U_\xi + \frac{2}{x} U_{\xi\eta} + u_{\eta\eta}$$

step 6: sub. in (2) we get,

$$A\left(\frac{y^2}{x^4} \cup_{\xi\xi} + \frac{2y}{x^3} U_\xi\right) + B\left(\frac{-y}{x^3} U_\xi \xi - \frac{y}{x^2} U_\xi \eta\right.$$
$$\left.- \frac{1}{x^2} \cup \xi\right) + c\left(\frac{1}{x^2} \cup \xi\xi + \frac{2}{x} \cup_{\xi\eta} + u_{\eta\eta}\right) + 0 = 0$$

$$x^2\left(\frac{y^2}{x^4} \cup_{\xi\xi} + \frac{2y}{x^3} U_\xi\right) + B\left(\frac{-y}{x^3} \cup_{\xi\xi} - \frac{y}{x^2} U_\xi \eta + y^2\left(\frac{1}{x^2} \cup_{\xi\xi} + \frac{2}{x} U_{\xi\eta} + U_{\eta\eta}\right) + 0 = 0\right.$$

$$\frac{y^2}{x^2} \cup_{\xi\xi} + \frac{2y}{x} \cup_\xi - \frac{2y^2}{x^2} \cup \xi\xi - \frac{2y^2}{x} \cup \xi\eta - \frac{2y}{x} \cup \xi + \frac{y^2}{x^2} \cup_{\xi\xi} + \frac{2y^2}{x} \cup \xi\eta + y^2 u_{\eta\eta} + 0 = 0$$

$$\frac{2y^2}{x^2} \cup_{\xi\xi} + \frac{2y}{x} \cup_\xi - \frac{2y^2}{x^2} \cup_{\xi\xi} - \frac{2y}{x} \cup_\xi + y^2 u_{\eta n} = 0$$

$$y^2 u_{\eta\eta} = 0$$

$$u_{\eta\eta} = 0$$

This is the canonical form of a parabolic equation.

Example 3:

Reduce the equation $U_{xx} + x^2 U_{yy} = 0$ to the canonical form.

Solution:

consider the eqn. $U_{xx} + x^2 U_{yy} = 0$ ... ... (1) compare this into the general PDE

$$AU_{xx} + BU_{xy} + CU_{yy} + DUx + EU_y + FU = G \ldots\ldots\ldots..(2)$$

Here $A = 1$ $B = 0$ $C = x^2$ $D = E = F = G = 0$

Step :1

$B^2 - 4AC = 0 - 4(1)x^2 = -4x^2 < 0$

∴ The equation is an elliptic everywhere except on the coordinate axis $x = 0$

step 2:

The characteristic equations are

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$
$$= \frac{0 \pm \sqrt{0 - 4(1)x^2}}{2(1)} = \frac{-\sqrt{-4x^2}}{2} = \frac{\pm i2x}{2}$$
$$= \pm ix$$
$$\frac{dy}{dx} = xi, \ \frac{dy}{dx} = -xi$$
$$dy = ixdx \ dy = -xidx$$

Step 3:

integrating

$$\int dy \quad = i\int xdx \qquad \int dy \quad = -i\int +xdx$$
$$y \quad = i\frac{x^2}{2} + c_1 \qquad y \quad = -i\frac{x^2}{2} + c_2$$
$$\Rightarrow 2y \quad = ix^2 = c_1 \qquad 2y + ix^2 = c_2$$

Step 4:

To transform the given equation to the canonical form

	Let $\xi = 2y - ix^2$	& $\eta = 2y + ix^2$
$\xi x = -i2x$	$\xi xx = -2i$	$\xi xy = 0$
$\xi y = 2$	$\xi yx = 0$	$\xi yy = 0$
$\eta x = i2x$	$\eta xx = 2i$	$\eta xy = 0$
$\eta y = 2$	$\eta yx = 0$	$\eta yy = 0$

Step: 5

$$U_x = U_\alpha \alpha x + U_\beta \beta_x$$
$$= U\alpha(0) + U_\beta(-2x) = -2xU_\beta$$

$$U_y = U_\alpha x + U_\beta \beta y$$
$$= U_\alpha(2) + U_\beta(0) = 2U_\alpha$$
$$U_{xx} = U_{\alpha\alpha} 2x^2 + 2U_{\alpha\beta}\alpha_x\beta_x + U_{\beta\beta}\beta x^2 + U_\alpha \alpha_x x + U_\beta \beta xx$$
$$= U_{\alpha\alpha}(0) + 2U_{\alpha\beta}(0)(-2x) + U_{\beta\beta}(-2x)^2 + U_\alpha(0) + U_\beta(-2)$$
$$= 4x^2 U_{\beta\beta} - 2U_\beta$$
$$U_{xy} = U_{\alpha\alpha}\alpha_x\,\alpha_y + U_{\alpha\beta}(\alpha_x\beta y + \alpha_y\beta x) + U_{\alpha\beta}\alpha_x + U_\alpha\alpha_{xy} + U_\beta \beta xy$$
$$= U_{\alpha\alpha}(0)(2) + U_{\alpha\beta}(0X0 + 2(-2x)) + U_{\alpha\beta}(0)(0) + U_\alpha(0) + 0$$
$$= -4x \cup \alpha\beta$$

$$U_{yy} = U_{\alpha\alpha}\alpha y^2 + 2U_{\alpha\beta}\alpha_y\ \ \beta y + U_{\beta\beta}\beta y^2 + U_\alpha \alpha yy + u_\beta \beta yy$$
$$= U_{\alpha\alpha}(y)^2 + 2U_{\alpha\beta}(u)(0) + U_{\beta\beta}(0) + U_\alpha(0) + U_\beta(0)$$
$$= 4U_{\alpha\alpha}$$

Step:6 sub in (2) we pet.

$$A(4x^2U_{\beta\beta} - 2U_\beta) + c(4U_{\alpha\alpha}) = 0$$
$$4x^2U_{\beta\beta} - 2U_\beta + x^2 4U_{\alpha\alpha} = 0$$
$$U_{\alpha\alpha} + u_{\beta\beta} = \frac{2u_\beta}{4x^2}$$
$$= \frac{1}{2x^2}U_\beta$$
$$U_{\alpha\alpha} + u_{\beta\beta} = \frac{-1}{2\beta}U_\beta$$

This the canonical form.

1.9. Equations with constant coefficients:

In the case of an equation with real constant coefficients, the equation is of the single type at all points in the domain. This is because the discriminant B^2-4AC is a constant.

From the characteristics equation

$$\frac{dy}{dx} = \frac{B+\sqrt{B^2-4AC}}{2A}$$

$$\frac{dy}{dx} = \frac{B-\sqrt{B^2-4AC}}{2A} \quad \ldots\ldots\ldots\ldots\ (1)$$

We can see that the characteristics

$$y = \left(\frac{B+\sqrt{B^2-4AC}}{2A}\right)x + c_1$$

$$y = \left(\frac{B-\sqrt{B^2-4AC}}{2A}\right)x + c_2$$

are two families of straight line. Consequently, the characteristic coordinates take the form

Let $\xi = y - \lambda_1 x,\ \ \eta = y - \lambda_2 x$ ………… (3)

Where $\lambda_{1,2} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$ ………… (4)

The linear second-order partial differential equation with constant coefficients may be written as

$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = G(x, y) \ldots\ldots\ldots..$ (5)

(A) Hyperbolic Equation

When $B^2 - 4AC > 0$, the equation is of hyperbolic type, in which case the characteristics form two distinct families.

Using equation (3),

Equation (5) becomes $u_{\xi\eta} = D_1 u_\xi + E_1 u_\eta + F_1 u + G_1(\xi, \eta)$ …………(6)

where D_1, E_1, and F_1 are constants. Here, since the coefficients are constants, the lower order terms are expressed explicitly.

When $A = 0$, Eq. (3.3.1) does not hold. In this case, the characteristic equation may be put in the form $-B(dx/dy) + C(dx/dy)^2 = 0$

which may again be rewritten as

$$dx/dy = 0$$
$$-B + C(dx/dy) = 0$$

Integration gives

$$x = c_1$$
$$x = (B/C)y + c_2$$

where c_1 and c_2 are integration constants. Thus, the characteristic coordinates are

$$\xi = x$$

Under this transformation, Eq. (3.3.5) reduces to the canonical form

$u_{\xi\eta} = D_1^* u_\xi + E_1^* u_\eta + F_1^* u + G_1^*(\xi, \eta)$ …………… (8)

where D_1^*, E_1^* and F_1^* are constants.

(B) Parabolic Equation

When $B^2 - 4AC = 0$, the equation is of parabolic type, in which case only one real family of characteristics exists. From Eqn (4), we find that

$\lambda_1 = \lambda_2 = (B/2A)$

so that the single family of characteristics is given by

$y = (B/2A)x + c_1$

where c_1 is an integration constant. Thus, we have

$\xi = y - (B/2A)x$

$\eta = hy + kx$ ………… (9)

where η is chosen arbitrarily such that the Jacobian of the transformation is not zero, and h and k are constants.

With the proper choice of the constants h and k in the transformation (9), Eq. (5) reduces to

$u_{\eta\eta} = D_2 u_\xi + E_2 u_\eta + F_2 u + G_2(\xi, \eta)$ ………….(10)

where D_2, E_2, and F_2 are constants.

If $B = 0$, we can see at once from the relation

$B^2 - 4AC = 0$

that C or A vanishes. The given equation is then already in the canonical form. Similarly, in the other cases when A or C vanishes, B vanishes. The given equation is then also in canonical form.

(C) Elliptic Equation

When $B^2 - 4AC < 0$, the equation is of elliptic type. In this case, the characteristics are complex conjugates.

The characteristic equations yield

$y = \lambda_1 x + c_1$

where λ_1 and λ_2 are complex numbers. Accordingly, c_1 and c_2 are allowed to take on complex values. Thus,

$\xi = y - (a + ib)x$

where $\lambda_{1,2} = a \pm ib$ in which a and b are real constants, and $a = B/2A$ and $b = \sqrt{4AC - B^2}/2A$.

Introduce the new variables

$$\alpha = \frac{1}{2}(\xi + \eta) = y - ax$$

Application of this transformation readily reduces Eq. (3.3.5) to the canonical form

$u_{\alpha\alpha} + u_{\beta\beta} = D_3 u_\alpha + E_3 u_\beta + F_3 u + G_3(\alpha, \beta)$ ………… (14)

where D_3, E_3 and F_3 are constants.

We note that $B^2 - AC < 0$, so neither A nor C is zero.

Example 1:

Consider the equation $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

Since $A = 4, B = 5, C = 1$, and $B^2 - 4AC = 9 > 0$, the equation is hyperbolic. Thus the characteristic equations take the form

$$\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{4}$$

and hence the characteristics are

The transformation

$$y = x + c_1$$
$$y = (x/4) + c_2$$
$$\xi = y - x$$
$$\eta = y - (x/4)$$

therefore, reduces the given equation to the canonical form

$$u_{\xi\eta} = \frac{1}{3}u_\eta - \frac{8}{9}$$

This is the first canonical form.

The second canonical form may be obtained by the transformation

$$\alpha = \xi + \eta$$
$$\beta = \xi - \eta$$

As $u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{3}u_\alpha - \frac{1}{3}u_\beta - \frac{8}{9}$

Example 2:

The equation $u_{xx} - 4u_{xy} + 4u_{yy} = e^y$

is parabolic since $A = 1, B = -4, C = 4$ and $B^2 - 4AC = 0$. Thus, we have from Eq. (9)

$$\xi = y + 2x$$
$$\eta = y$$

in which η is chosen arbitrarily. By means of this mapping, the equation transforms into

$$u_{\eta\eta} = \frac{1}{4}e^\eta$$

Example 3:

Consider the equation $u_{xx} + u_{xy} + u_{yy} + u_x = 0$

Since $A = 1, B = 1, C = 1$, and $B^2 - 4AC = -3 < 0$, the equation is elliptic.

We have

$$\lambda_{1,2} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

and hence,

$$\xi = y - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x$$

$$\eta = y - \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)x$$

Introducing the new variables

$$\alpha = \frac{1}{2}(\xi + \eta) = y - \frac{1}{2}x$$

$$\beta = \frac{1}{2i}(\xi - \eta) = -\frac{\sqrt{3}}{2}x$$

the given equation is then transformed into canonical form

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{2}{3}u_\alpha + \frac{2}{\sqrt{3}}u_\beta$$

Example 4:

Consider the wave equation $u_{tt} - c^2 u_{xx} = 0,\ c$ is constant

Since $A = -c^2, B = 0, C = 1$, and $B^2 - 4AC = 4c^2 > 0$, the wave equation is hyperbolic everywhere. According to (3.2.4), the equation of characteristics is

$$-c^2\left(\frac{dt}{dx}\right)^2 + 1 = 0 \text{ or } dx^2 - c^2 dt^2 = 0$$

Therefore $\quad x + ct = \xi =$ constant

$\qquad\qquad\quad x - ct = \eta =$ constant

Thus the characteristics are straight lines, which are shown in Fig. 3.3.1. The characteristics form a natural set of coordinates for a hyperbolic equation.

In terms of new coordinates ξ and η defined above, we obtain

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{tt} = c^2\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right)$$

so that the wave equation becomes

$$-4c^2 u_{\xi\eta} = 0$$

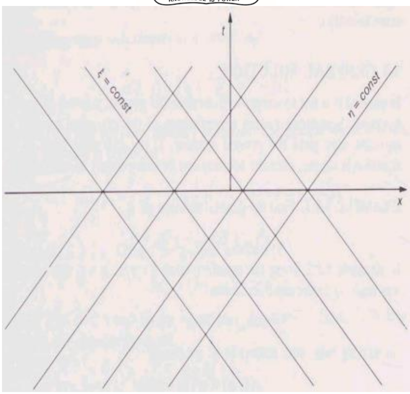

Figure 1. Characteristics for the wave equation.

Since $c \neq 0$, we have

$u_{\xi\eta} = 0$

Integrating with respect to ξ, we obtain $u_\eta = \psi_1(\eta)$

where ψ_1 is the arbitrary function of η. Integrating again with respect to η, we obtain

$u(\xi, \eta) = \int \psi_1(\eta) d\eta + \phi(\xi)$

If we set $\psi(\eta) = \int \psi_1(\eta) d\eta$, the solution is

$u(\xi, \eta) = \phi(\xi) + \psi(\eta)$

which is, in terms of the original variables x and t,

$u(x, t) = \phi(x + ct) + \psi(x - ct)$

provided ϕ and ψ are arbitrary but twice differentiable functions.

Note that ϕ is constant on 'wavefronts' $x = -ct + \xi$ that travel toward decreasing x, as t increases, whereas ψ is constant on wavefronts $x = ct + \eta$ that travel toward increasing x as t increases. Thus, any general solution can be expressed as the sum of two waves, one traveling to the right with constant velocity c and the other traveling to the left with the same velocity c.

1.10. General Solution:

In general, it is not so simple to determine the general solution of a given equation. Sometimes further simplification of the canonical form of an equation may yield the general solution. If the canonical form of the equation is simple, then the solution can be immediately ascertained.

Example 1:

Find the general solution of $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0$

using the transformation $\xi = y/x, \eta = y$, this equation was reduced to the canonical form $u_{\eta\eta} = 0$ for $y \neq 0$

Integrating twice with respect to η, we obtain $u(\xi, \eta) = \eta f(\xi) + g(\xi)$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions. In terms of the independent variables x and y, we have $u(x, y) = yf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$

Example 2:

Determine the general solution of $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

Using the transformation $\xi = y - x, \eta = y - (x/4)$, the canonical form of this equation is $u_{\xi\eta} = \frac{1}{3}u_\eta - \frac{8}{9}$

By means of the substitution $v = u_\eta$, the preceding equation reduces to

$$v_\xi = \frac{1}{3}v - \frac{8}{9}$$

This can be easily integrated by separating the variables. Integrating with respect to ξ, we have

$$v = \frac{8}{3} + \frac{1}{3}e^{(\xi/3)}F(\eta)$$

Integrating again with respect to η, we obtain

$$u(\xi, \eta) = \frac{8}{3}\eta + \frac{1}{3}g(\eta)e^{\xi/3} + f(\xi)$$

where $f(\xi)$ and $g(\eta)$ are arbitrary functions. The general solution of the given equation is therefore

$$u(x, y) = \frac{8}{3}\left(y - \frac{x}{4}\right) + \frac{1}{3}g\left(y - \frac{x}{4}\right)e^{\frac{1}{3}(y-x)} + f(y - x)$$

Example 3:

Obtain the general solution of $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$

Since $B^2 - 4AC = 64 > 0$, the equation is hyperbolic. Thus, from Eqs. (3.3.2), the characteristics are

$$y = 3x + c_1$$
$$y = \frac{1}{3}x + c_2$$

Using the transformation

$$\xi = y - 3x$$
$$\eta = y - \frac{1}{3}x$$

the given equation is reduced to

$$\frac{64}{3}u_{\xi\eta} = 0$$

Hence, we obtain $u_{\xi\eta} = 0$

Integration yields $u(\xi,\eta) = f(\xi) + g(\eta)$

In terms of the original variables, the general solution is $u(x,y) = f(y - 3x) + g\left(y - \frac{x}{3}\right)$

Exercises:

1. Determine the region in which the given equation is hyperbolic, parabolic, or elliptic, and transform the equation in the respective region to the canonical form.

   a. $xu_{xx} + u_{yy} = x^2$

   b. $u_{xx} + y^2u_{yy} = y$

   c. $u_{xx} + xyu_{yy} = 0$

   d. $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = e^x$

   e. $u_{xx} + u_{xy} - xu_{yy} = 0$

   f. $e^xu_{xx} + e^yu_{yy} = u$

UNIT- II

Cauchy Problem: The Cauchy problem – Cauchy-Kowalewsky theorem – Homogeneous wave equation – Initial Boundary value problem- Non-homogeneous boundary conditions – Finite string with fixed ends – Non-homogeneous wave equation – Riemann method – Goursat problem – spherical wave equation – cylindrical wave equation.

Chapter 2: Sections 2.1 to 2.11

2.1. The Cauchy Problem.

consider a second order partial differentiation equation for the function u in the independent variables x and y and suppose that this equation can be solved explicitly for u_{yy} and hence can represented in the form.

$u_{yy} = F(x, y, u_1 u_x, g_y u_y, u_{xx}, u_{yy})$

For some value $y = y_0$, we prescribe the initial values of the unknown function and of the derivative with respect to y.

u $(x, y_0) = f(x)$
$u_y(x, y_0) = g(x)$.

The problem of determining the solution of equation satisfying the initial conditions is known as initial value problem.

Initial - value problem of a vibrating string:

The curve equation, $u_{tt} = c^2 u_{xx}$ satisfying the initial conditions.

$u(x, t_0) = u_0(x)$.
$u_t(x, t_0) = v_0(x)$.

where $u_0(x)$ is the initial displacement and $v_0(x)$ is the initial velocity.

Note: Cauchy problem:

In initial value problems, the initial values usually refer to the data assigned at $y = y_0$. It is not essential that these values be given along the line $y = y_0$, they may very well be prescribed along some curve L_0 in the xy plane. In such a context, the problem is called the Cauchy problem instead of the initial value problem.

In general, let us consider the equation.

$Au_{xx} + Bu_{xy} + Cu_{yy} = F(x, y, u_1 u_x, u_y) \to (*)$

where A, B & C are functions of x & y.

Let (x_0, y_0) denote points on a smooth curve L_0 in the $x - y$ plane. Let the parametric equation of this curve L_0 be $x_0 = x_0(\lambda),\ y_0 = y_0(\lambda)$.

where λ is a parameter.

Two function $f(\lambda)$ & $g(\lambda)$ are proscribed along the curve L_0.

$$u = f(\lambda).$$
$$\frac{\partial u}{\partial n} = g(\lambda).$$

The functions $f(\lambda)$ & $g(\lambda)$ are called the Cauchy data.

The solution of the Cauchy problem is a surface called an integral surface.

$\Rightarrow a = f(\lambda)$

diff w.r.t ' λ '.

$$\Rightarrow \frac{du}{d\lambda} = \frac{\partial u}{\partial x}\frac{dx}{d\lambda} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{df}{d\lambda} \quad \ldots\ldots\ldots (1)$$

Now $\frac{dy}{dn} = g(\lambda)$

Expand this $\Rightarrow \frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\frac{dx}{dn} + \frac{\partial u}{\partial y}\frac{dy}{dn} = g.$ .... (2)

But $\frac{dx}{dx} = -\frac{dy}{ds}$ and $\frac{dy}{dn} = \frac{dx}{ds}$.

$$\Rightarrow \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}\frac{dy}{ds} + \frac{\partial u}{\partial y}\frac{dx}{ds} = g.$$

From equation (2) $\begin{vmatrix} \frac{dx}{d\lambda} & \frac{dy}{d\lambda} \\ -\frac{dy}{ds} & \frac{dx}{ds} \end{vmatrix} = \frac{dx}{d\lambda}\cdot\frac{dx}{ds} + \frac{dy}{d\lambda}\frac{dy}{ds}.$

$$= \frac{(dx)^2 + (dy)^2}{ds d\lambda} \neq 0.$$

Equation (1) diff. w.r.t. x. $\frac{\partial^2 u}{\partial x^2}\frac{dx}{d\lambda} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{dy}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial u}{\partial x}\right)$ .... (3)

Equation (1) diff. w.r.t y.

$$\frac{\partial^2 u}{\partial x\,\partial y}\frac{dx}{\partial\lambda} + \frac{\partial^2 u}{\partial y^2}\frac{dy}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial u}{\partial y}\right) \rightarrow (4)$$

$$(*) \Rightarrow A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = F \rightarrow (5)$$

From equation 3,4,5.

$$\begin{vmatrix} \frac{dx}{d\lambda} & \frac{dy}{d\lambda} & 0 \\ 0 & \frac{dx}{d\lambda} & \frac{dy}{d\lambda} \\ A & B & C \end{vmatrix} = \frac{dx}{d\lambda}\left[C\frac{dx}{d\lambda} - B\frac{dy}{d\lambda}\right] -$$

$$\frac{dy}{d\lambda}\left[0 - A\frac{d\infty}{d\lambda}\right] + 0.$$

$$= c\left(\frac{dx}{d\lambda}\right)^2 - B\frac{dx}{d\lambda}\frac{dy}{d\lambda} + A\left(\frac{dy}{d\lambda}\right)^2$$

$$\neq 0.$$

$\div$ by $\left(\frac{dx}{d\lambda}\right)^2$,

$$C - B\frac{\frac{dy}{d\lambda}}{\frac{dy}{dx}} + A\frac{\left(\frac{dy}{d\lambda}\right)^2}{\left(\frac{dx}{d\lambda}\right)^2} = 0.$$

$$C - B\frac{dy}{dx} + A\left(\frac{dy}{dx}\right)^2 = 0.$$

$$\Rightarrow A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0.$$

which is the characteristic equation.

If Equation (*) and functions $f(\lambda)$ & $g(\lambda)$ are analytic. The solution is

$$u(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!\,(n-k)!}\frac{\partial^n u_0}{\partial x_0^k\,\partial y_0^{n-k}}(x-x_0)^k(y-y_0)^{n-k}.$$

2.2. Cauchy-Kowalewsky Theorem.

Let the partial differential equation be.

$$u_{yy} = F\big(y, x_1, x_2, \dots, x_n, u_1 u_y, u_{x_1}, u_{x_2}, \dots, u_{x_n},$$
$$u_{x,y}, u_{x_2 y}, \dots, u_{x_n y}, u_{x_1 x_1}, u_{x_2 x_2}, \dots, u_{x_n x_n}\big).$$

Let the initial conditions.

$$u = f(x_1, x_2, \dots, x_n).$$
$$u_y = g(x_1, x_2, \dots, x_n)$$

be given on the non-characteristic manifold $y = y_0$.

2.3. Homogeneous Wave Equation:

Cauchy problem of an infinite string with initial condition.

$u_{tt} - c^2 u_{xx} = 0$ … … …. (1)

Initial condition, $u(x,0) = f(x)$ ……….. (2)

$u_t(x,0) = g(x)$ ………… (3)

Here $A = -c^2, B = 0, C = 1.$
$\therefore B^2 - 4AC = 4c^2 > 0.$

(i.e.) The given equation is Hyperbolic ∀ the domain

Now

$$\frac{dE}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$
$$= \pm\frac{\sqrt{4c^2}}{2(-c^2)} = \pm\frac{2c}{2c^2}$$
$$= \frac{1}{c}, -\frac{1}{c}.$$

$\mathrm{dt} = \frac{1}{c}dx.$

$t = \frac{1}{c}x + c_1$ $\qquad$ $dt = -\frac{1}{c}dx.$

$ct - x = c_1.$ $\qquad$ $t = -\frac{1}{c}x + c_2$

$ct + x = c_2.$

Introducing the characteristic coordinates

$\xi = ct + x \quad \xi_x = 1 \quad \eta_x = 1$
$\eta = x - ct \quad \xi_t = c \quad \eta_t = -c$
$\xi_{xx} = 0 \quad \eta_{tt} = 0$
$\xi_{tt} = 0 \quad \eta_{xt} = 0$

$u_{xt} = u_{\xi\xi} + 2u_{\xi\eta\eta} + u_{\eta\eta}$ ………… (4)

$u_{tt} = u_{\xi\xi}(c)^2 + 2u_{\xi\eta}(c)c - c) + u_{\eta\eta}(-c)^2$

$= c^2[u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}]$ ………….. (5)

$u_{tt} - c^2u_{xx} = 0.$

$c^2u_{\xi\xi} - c^2u_{\xi\eta} + u_{\eta\eta}{}^2 - cu_{\xi\xi} - 2c^2u_{\xi\eta} - 2u_{\eta\eta} = 0$

$-4c^2u_{\xi\eta} = 0$

$u_{\xi\eta} = 0$ ($\because c \neq 0$) ………… (6)

Integrating w.r.t ξ,

$$\int \frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial n}\right)d\xi = 0$$

$\frac{\partial u}{\partial\eta} = \psi^*(\eta)$. where $\psi^*(\eta)$ is arbitrary function of η.

Integrating w.r.to η,

$$\int u_\eta d\eta = \int \psi^*(\eta) d\eta$$

$$u = \psi(\eta) + \phi(\xi).$$

$$\therefore u(\xi, \eta) = \phi(\xi) + \psi(\eta).$$

where φ and ψ are arbitrary function.

Transforming to the original variable x and t, we find the general solution of wave equation,

$$u(x,t) = \phi(x + ct) + \psi(x - ct) \qquad \ldots\ldots\ldots (7)$$

provided, ϕ and ψ are twice differentiable function.

Substituting equation (2) in (7),

$$f(x) = u(x, 0) = \phi(x) + \psi(x) \quad \ldots\ldots (8)$$

Substituting equation (3) in (9)

$$u_t(x,t) = c\phi'(x + ct) - c\psi(x - ct) \qquad \ldots\ldots\ldots (9)$$

$$g(x) = u_t(x,t) = c\phi'(x) - c\psi'(x) \qquad \ldots\ldots\ldots. (10)$$

From equation (10) $\Rightarrow$

$$c\phi'(x) - c\psi'(x) = g(x)$$

$$\Rightarrow \phi'(x) - \psi'(x) = \frac{1}{c} g(x)$$

Integrating, w.r.to x, $\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(\tau)d\tau + k$ ………(11)

where x_0 & k are

Solving equation (9) & (11) we get.

$$(9) + (11) \Rightarrow 2\phi(x) = f(x) + \frac{1}{c}\int_{x_0}^{x} g(\tau)d\tau + k.$$

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{\alpha_0}^{x} g(\tau)d\tau + \frac{k}{2} \ldots\ldots\ldots\ldots\ldots (12)$$

From equation (9) - (11)

$$\Rightarrow 2\psi(x) = f(x) - \frac{1}{c}\int_{x_0}^{x} g(\tau)d\tau - \frac{1}{c}$$

$$= \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\tau)d\tau - \frac{k}{2}. \ldots\ldots\ldots\ldots(13)$$

Substituting (12), (13) in (7).

$$u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{0x}^{x+ct} g(\tau)d\tau + \frac{k}{2} + \frac{1}{2}f(x_1+ct)$$

$$-\frac{1}{2c}\int_{x_0}^{x-ct} g(\tau)d\tau - \frac{k}{2}$$

$$= \frac{1}{2}[f(x+ct)+f(x-ct)] +$$

$$\frac{1}{2c}\left[\int_{x_0}^{x+ct} g(\tau)d\tau - \int_{x_0}^{x-ct} g(\tau)d\tau\right]$$

$$= \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\left[\int_{x_0}^{x+ct} g(\tau)d\tau + \int_{x-ct}^{x_0} g(\tau)_d\right)$$

$$= \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau \overset{x}{\underset{x}{\rightarrow}} \quad \ldots\ldots\ldots.(14)$$

This is called the d'Alembert solution of the Cauchy problem for the one dimensional wave equation.

Exercises 1:

a) $\boldsymbol{u_{tt} - c^2u_{xx} = 0, u(x,0) = 0, u_t(x,0) = 1}$

Solution:

$u_{tt} - c^2u_{xx} = 0$ ………… (1)

$u(x,0) = 0$ ……….. (2)

$u_t(x,0) = 1$ …………… (3)

$$B^2 - 4Ac = -c^2, B = 0, C = 1.$$

$$= 4c^2 > 0. \text{ (Hyperbolic)}$$

$$\frac{dy}{dx} = \frac{0 \pm \sqrt{4c^2}}{-2c^2} = \pm\frac{2c}{-2c^2} = \pm\frac{1}{c}.$$

$$\frac{dy}{dx} = \frac{1}{c} \Rightarrow dy_t = \frac{1}{c}dx$$

$$ty = \frac{1}{c}x + c_1.$$

$c_4 = x + c_1.$

$$\frac{dy}{dx} = -\frac{1}{c} \Rightarrow cdy = -dx$$

$$cy = -x + c_2$$

$$x + cy = c_2.$$

$$\Rightarrow c_1 = x + ct.$$

$$c_2 = x - ct.$$

Introducing the new variable,

$\xi = x + ct.$ $\xi_x = 1$ $\eta_x = 1$

$\eta = x - ct$ $\xi_{xx} = 0$ $\eta_{xx} = 0$

$u_x = u_\xi \xi_x + u_\eta \eta_x$ $\xi_t = c$ $\eta_t = -c$

$u_t = u_\xi \xi_t + u_\eta \eta_t$ $\xi_{tt} = 0$ $\eta_{tt} = 0$

$$u_{xx} = u_\xi \xi_{xx} \quad + u_{\xi\xi} \xi_x \xi_x + u_\eta \eta_{xx} + \eta_x u_{\eta\eta} \eta_x.$$
$$+ \xi_x u_{\xi\eta} \eta_x + \eta_x u_{\eta \xi_t} \xi_x.$$

$=u_{\xi\xi}$+2uu$_{\xi\eta} + u_{\eta\eta}$ ……..(4)

u_{tt}=c^2[$u_{\xi\xi}$+2uu$_{\xi\eta} + u_{\eta\eta}$] ……….(5)

Substituting in (1),

$-4c^2 u_{\xi\eta} = 0.$

$u\xi\eta = 0\ (\because c \neq 0) \to (6).$

Integrating, $w \cdot r \cdot$ to ξ,

$u_n = \psi^m(\eta)$

Integrating $w \cdot r \cdot$ to n,

$u(\varepsilon, \eta) = \psi(\eta) + \phi(\xi).$

$u(x,t) = \phi(x + ct) + \psi(x - ct)$ …………….. (7)

Now sub (2) in (7), we get

$u(x, 0) = \phi(x) + \psi(x) = 0$ …………. (8)

$u_t(x, \text{t}) = \phi'(x + ct)(c) - c\psi'(x - ct)$ …………. (9)

Substituting (3) in (9), $u_t(x, 0) = \phi'(x)c - c\psi'(x) = 1$ ………(10)

⇒ Integrating $w \cdot r \cdot to$, t

$\phi'(x) - \psi'(x) = \frac{1}{c} g(x) \cdot 1$

$\phi(x) - \psi(x) = \frac{1}{c} \int_{x_0}^{x} d\tau + k$ …………. (11)

(8)+ (11) : $2\phi(x) = \frac{1}{c} \int_{x_0}^{x} d\tau + k$

(8) - (11) $2\psi(x) = -\frac{1}{c} \int_{x_0}^{x} d\tau - k.$ ……..(13)

(12), (13) in (7)

$$u(x,t) = \frac{1}{2}\left(\frac{1}{c}\int_{x_0}^{x+ct} d\tau + k\right) + \frac{1}{2}\left(-\frac{1}{c}\int_{x_0}^{x-ct} d\tau - k\right)$$

$$= \frac{1}{2c}\int_{x_0}^{x+ct} d\tau + \frac{k}{2} - \frac{1}{2c}\int_{\alpha_0}^{x-ct} d\tau - \frac{1c}{2}.$$

$$= \frac{1}{2c}\int_{x_0}^{x+ct} d\tau + \frac{1}{2c}\int_{x-ct}^{x_0} d\tau.$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} d\tau.$$

$$= \frac{1}{2c}[x + ct - x + ct]$$

$$= \frac{1}{2c} \times 2ct = t.$$

b.) $u_{tt} - c^2 u_{xx} = 0, u(x,0) = \sin x, u_t(x,0) = x^2$.

Solution:

$u_{\xi n} = 0.$

$u(x,t) = \phi(x + ct) + \psi(x - ct).$

$u_t(x,t) = c\phi'(x + ct) - c\psi'(x - ct).$

$\Rightarrow u(x,0) = \phi(x) + \psi(x) = \sin x \ldots\ldots\ldots (1)$

$u_t(x,0) = c\phi'(x) - c\psi'(x) = x^2.$

$\phi'(x) - \psi'(x) = \frac{1}{c}x^2.$

$\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} \tau^2 d\tau + k \quad \ldots\ldots\ldots (2)$

$$(1) + (2) \Rightarrow \; Q\phi(x) = \sin x + \frac{1}{2}\int^{x} \tau'\,\tau' d\tau + k$$

$$(1) - (1) \Rightarrow 2\psi(x) = \sin x - \frac{1}{c}\int_{x_0}^{x_0} \tau' d\tau - k$$

$$\Rightarrow u(x,t) = \frac{1}{2}\sin(x + ct) + \frac{1}{2c}\int_{x_0}^{x+ct} \tau^2 d\tau + \frac{k}{2} +$$

$$\frac{1}{2}\sin(x - ct) - \frac{1}{2c}\int_{x_0}^{x-ct} \tau^2 d\tau - \frac{\mathrm{k}}{2}.$$

$$= \frac{1}{2}[\sin(x + ct) + \sin(x - ct)] + \frac{1}{2c}\int_{\mathrm{x-ct}}^{x+ct} \tau^2 d\tau.$$

$$= \frac{1}{2}\left[2\ \frac{\sin(x+ct)+\sin(x-ct)}{2}\ \ \frac{\cos(x+ct-x+ct)}{2}\right] + \frac{1}{2c}\int_{x\_ct}^{x+ct} \tau^2 d\tau.$$

$$= \frac{1}{2} \times 2\left[\sin\frac{2x}{2}\cos\frac{2ct}{2}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tau^2 dt.$$

$$= \sin x \cos 2t + \frac{1}{2c}\int_{x-ct}^{x+ct} \tau^2 d\tau.$$

$$= \sin x \cos ct + \frac{1}{2c}\left[\frac{\tau^3}{3}\right]_{x-ct}^{x+ct}.$$

$$= \sin x \cos ct + \frac{1}{2c}\left[\frac{(x+ct)^3-(x-ct)^3}{3}\right] \quad \ldots\ldots\ldots..(3)$$

$a^3 - b^3 = (a-b)(a^2+b^2+ab).$

Take $(x+ct)^3 - (x-ct)^3 = [x+ct-x+ct][(x+ct)^2 +$
$(x-ct)^2 + (x+ct)(x-ct)].$

$= (2ct)[x^2 + 2ctx + c^2t^2 + x^2 - 2ctx + c^2t^2$
$+x^2 - xct + xc' - c^2/t^2].$

$= 2ct \times [3x^2 + c^2t^2].$

Substituting equation in (3)

$$= \sin x \cos ct + \frac{1}{2c} \times 2\text{ct}\left[\frac{3x^2+c^2t^2}{3}\right].$$

$$= \sin x \cos ct + \text{t}x^2 + \frac{c^2t^3}{3}$$

c.) $u_{tt} - c^2 u_{xx} = 0, u(x,0) = x^3\ u_t(x,0) = x.$

Solution:

$u(x,t) = \phi(x+ct) + \phi(x-ct).$
$u(x,0) = \phi(x) + \phi(x) = x^3 \to (1)$
$$\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} \tau \, d\tau + k \dots\dots\dots (2).$$
$$(1) + (2) \Rightarrow 2\phi(x) = x^3 + \frac{x}{c}\int_{x_0}^{x} \tau d\tau + k.$$

$$(1) - (2) \Rightarrow 2\psi(x) = x^3 - \frac{1}{c}\int_{x_0}^{c} \tau d\tau - k$$
$$u(x,t) = \frac{(x+ct)^3}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} \tau d\tau + \frac{k}{2} + \frac{x^3 - ct}{2} - \frac{1}{2c}\int_{x_0}^{x-ct} \tau dt$$
$$= \frac{1}{2}[(x+ct)^3 + (x-ct)^3] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tau d\tau.$$
$a^3 + b^3 = (a+b)(a^2 + b^2 - ab).$
$$= \frac{1}{2}[2x^3 + 6x\,c^2t^2] + \frac{1}{2c}\left[\frac{\tau^2}{2}\right]_{x-ct}^{x+ct}.$$
$$= \frac{1}{2} \times 2[x^3 + 3\text{x}c^2t^2] + \frac{1}{2c}[(x+ct)^2 - (x-ct)^2]$$
$$= (x^3 + 3xc^2t^2) + \frac{1}{4c}4xct.$$
$$= x^3 + 3xc^2t^2 + 2xt.$$

d.) $\boldsymbol{u_{tt} - c^2 u_{xx} = 0, u(x,0) = \cos x, u_t(x,0) = e^{-1}.}$

Solution:

$u(x,t) = \phi(x+ct) + \psi(x-ct).$
$u(x,0) = \phi(x) + \psi(x) = \cos x \dots\dots\dots.(1)$
$$\phi(x) - \psi(x) = \frac{1}{c}e^{-1}\int_{x_0}^{x} d\tau + k \to (2)$$

$$(1)+(2) \Rightarrow 2\phi(x) = \cos x + \frac{1}{c}e^{-1}\int_{x_0}^{x} d\tau + k$$

$$(1) - (2) \Rightarrow 2\psi(x) = \cos x - \frac{1}{c}e^{-1}\int_{x_0}^{x_0} d\tau - k.$$

$$u(x,t) = \frac{\cos(x+ct)}{2} + \frac{1}{2c}e^{-1}\int_{x_0}^{x+ct} d\tau + \frac{\mathrm{k}}{2} +$$

$$\frac{\cos(x-ct)}{2} - \frac{1}{2c}e^{-1}\int_{x_0}^{x-ct} d\tau - \frac{\mathrm{k}}{2}$$

$$= \frac{1}{2}[\cos(x+ct)+\cos(x-ct)] + \frac{1}{2}e^{-1}\int_{x-ct}^{x+ct} d\tau.$$

$$= \frac{1}{2}\left[2\cos\left(\frac{x+ct-x+ct}{2}\right)\cos\left(\frac{x+ct+x-ct}{2}\right)\right].$$

$$+\frac{1}{2c}[\tau]_{x-ct}^{x+ct} \times e^{-1}$$

$$= \cos ct \cos x + \frac{1}{2c}[x+ct-x+ct] \times e^{-1}.$$

$$\cos A + \cos B = 2\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right).$$

$$= \cos \mathrm{ct} \cos x + \frac{1}{2c} \times 2ct \times e^{-1}$$

$$= \cos c\,t \cos x + t/\mathrm{e}$$

e.) $u_{tt} - c^2 u_{xx} = 0,\ u(x,0) = \log(1+x^2),\ u_t(x,0) = 2.$

Solution:

$u(x,0) = \phi(x) + \psi(x) = \log(1+x^2)$ …….. (1).

$\phi(x) - \psi(x) = \frac{2}{c}\int_{x_0}^{x} dt + k$ ………. (2)

$$(1)+(2) \Rightarrow 2\phi(x) = \log(1+x^2) + \frac{2}{c}\int_{x_0}^{x} dt + k$$

$$(1)-(2) \Rightarrow 2\psi(x) = \log(1+x^2) - \frac{2}{c}\int_{x_0}^{x} d\tau - k.$$

$$u(x,t) = \frac{\log[1+(x+ct)^2)}{2} + \frac{2}{2c}\int_{x_0}^{x+ct} d\tau + \frac{\mathrm{k}}{2} - \frac{\mathrm{k}}{2} + \frac{\log[1+(x-ct)^2)}{2} + \frac{2}{2c}\int_{x_0}^{x-ct} d\tau$$

$$= \frac{1}{2}[\log[1 + x^2 + 2xct + c^2t^2] + \log[1 + x^2 - 2xct + c^2t^2]$$
$$+ \frac{2}{c}\int_{x-ct}^{x+ct} d\tau$$
$$= \frac{1}{2}[\log[1 + x^2 + 2xct + c^2t^2] + \log[1 + x^2 - 2xct + c^2t^2] + \frac{2}{c}[x + ct - x + ct]$$
$$= \frac{1}{2}\{\log(1 + x^2 + 2cxt + c^2t^2) + \log(1 + x^2 - 2xct + c^2 + 1)\} + 2t.$$

f) $u_{tt} - c^2u_{xx} = 0, u(x, 0) = x, u_t(x, 0) = \sin x.$

Solution:

$u(x, t) = \phi(x + ct) + \psi(x - ct).$

$u(x, 0) = \phi(x) + \phi(x) = x \ldots\ldots..(1)$

$\phi(x) - \phi(x) = \frac{1}{c}\int_{x_0}^{x} \sin\tau d\tau$+k ……. (2)

$$(1) + (2) 2\psi(x) = x + \frac{1}{c}\int_{x_0}^{x} \sin\tau d\tau + \text{k}.$$

$$(1) - (2) \Rightarrow 2\phi(x) = x - \frac{1}{c}\int_{x_0}^{x} \sin\tau d\tau - k$$

$$u(x, t) = \frac{(x + ct)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} \sin\tau d\tau + \frac{kt}{2} + \frac{(x - ct)}{2}$$
$$- \frac{1}{2c}\int_{x_0}^{x-ct} \sin\tau d\tau - \frac{k}{2}$$
$$= \frac{1}{2}[x + ct + x - ct] + \frac{1}{2c}[-\cos\tau]^{x+ct}.$$
$$= \frac{2x}{2} + \frac{1}{2c}[-(\cos(x + ct) + \cos(x - ct)]$$
$$= x + \frac{1}{2c}[-\cos(x - ct) + \cos(x + ct)].$$
$$= x + \frac{1}{2c}\left[x\sin\left(\frac{x - ct + x + ck}{2}\right)\sin\left(\frac{x + ct - x - ct}{2}\right]\right.$$
$$= x + \frac{1}{c}\sin x \text{ sinct}.$$

Note:

The solution $u(x, t)$ depends only on the initial values at pts btwn $x - ct$ and $x + ct$-and not at all initial values outside. this interval on the line $t = 0$. This interval is called the domain of dependence of the variables (x, t).

Note:

For every $\epsilon > 0$ and for each time interval $0 \leq t \leq t_0$, fr a number $\delta(\varepsilon, t_0)$ sot

$|u(x,t) - u^*(x,t)| < \varepsilon.$

whenever,

$|f(x) - f^*(x)| < \delta, \ |g(x) - g^*(x)| < \delta.$

From the equation, $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau,$

we have.

$$|u(x,t) - u^*(x,t)| \leq \frac{1}{2}|f(x+ct) - f^*(x+ct)| + \frac{1}{2}\Big| f(x-ct) - f^*(\mathrm{x-ct})$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} |g(x) - g^*(x)|d\tau < \ \varepsilon \quad ,where \ \varepsilon = \delta(1+t_0)$$

For any finite time interval $0 < t < t_0$, a small change in the initial data only produces a small change in the solution.

Example 1:

Find the solution of the initial value problem

$$u_{tt} - c^2 uxx ,u(x,0) = \sin x, \ u_t(x,0) = \cos x.$$

$$u(x,t) = \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos\tau d\tau.$$

$$= \sin x \cos ct + \frac{1}{2c}[\sin(x+ct) - \sin(x-ct)].$$

$$= \sin x \cos ct + \frac{1}{c}\cos x \sin ct.$$

Note: 1.

If an initial displacement or an initial velocity is located in a small neighborhood of point (x_0, t_0). It can influence only the area $t > t_0$, bounded by two characteristics $x - ct =$ constant and $x + ct =$ constant with slope $\pm\frac{1}{c}$ passing through the (x_0, t_0). This means that the initial displacement propagates at the speed c, whereas the effect of the initial velocity propagates at all speeds upto c.This infinite sector is called the domain of influence of the point (x_0, t_0).

Note: 2 According to the equation $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x+ct}^{x+ct} g(\tau)d\tau,$ the value of $u(x_0, t_0)$ depends on y initial data f and g in the interval $[x_0 - ct_0, x_0 + ct_0\ ]$ initial data f and g in the interval $[x_0 - \mathrm{c}t_0\ , x_0 + ct_0]$ which is cut out of the initial line by the two characteristic $x - ct =$ constant & $x + ct =$ constant with slope $\pm\frac{1}{c}$ passing through the $point$ (x_0, t_0). The interval $[x_0 - ct_0, x_0 + ct_0]$ on the lire $t = 0$ is called the domain of dependence.

The physical significance of the d' Alembert solution.

$u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(\tau)d\tau + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x+ct} g(\tau)d\tau,$

(or) $u(x,t) = \phi(x+ct) + \psi(x-ct)$.

$$\phi(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\xi} g(\tau)d\tau.$$

$$\psi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)d\tau.$$

$\phi(x + (t)$ represents a progressive above travelling in the negative x-direction with speed c without change of shape.

Similarly, $\psi(x - ct)$ is also a progressive wave propagating in the positive x-direction with the same speed c. without change of shape. At $t = 0$, the shape of this function is $u = \phi(x)$. At a subsequent time its shape is given by $u = \psi(x - ct)$, (or) $u = \psi(\xi)$ where $\xi = x - ct$ is the new coordinate obtatied by translating the origin a distance ct to the right. Thus the shape of the curve remains the same as time progresses, but moves to the right with velocity c.

$u(x,t) = \sin(x \mp ct)$

represent sinusoidal waves travelling with speed c in the +ve and - ve directions respectively without change of shape.

To interpret the d'Alembert formula, we consider two cases:

case (i):

when the initial velocity zero, is $g(x) = 0$.

Then the d'Alembert solution has the form

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)].$$

Now suppose the initial displacement $f(x)$ is different from zero in an interval $(-b, b)$. Then in this case the forward and the backward waves are represented by,

$$u = \frac{1}{2}f(x).$$

2.4. Initial-Boundary value problems:

(A) Semi-Infinite string with a fixed End:

consider a semi-infinite vibrating strong a fixed end, is.

$$u_{tt} = c^2 u_{xx},\ 0 < x < \infty, \qquad t > 0$$
$$u(x,0) = f(x),\ 0 \le x < \infty,$$
$$u_t(x,0) = g(x),\ 0 \leqslant x < \infty.$$
$$u(0,\mathrm{t}) = 0,1)\ 0 \le t < \infty.$$
$$w \cdot k \cdot t\ u(x,t) = \phi(x+ct) + \psi(x-ct).$$

where $\phi(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\xi} g(\tau)d\tau + \frac{k}{2}$ ……….(1)

$$\psi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)d\tau - \frac{k}{2} \quad ……….(2)$$
$$u(0,t) = \phi(ct) + \psi(-ct) = 0.$$
$$\Rightarrow \psi(-ct) = -\phi(ct)$$

Let $\alpha = -ct.$

$\psi(\alpha) = -\phi(-\alpha).$

Replacing α by $x - ct$, we obtain for $x < ct.$ $\psi(x-ct) = -\phi(ct-x)$ ……….. (3)

using $\phi(\xi), (i.e.)$ equation (1).

(3) becomes, $\psi(x-ct) = -\frac{1}{2}f(ct-x) - \frac{1}{2c}\int_0^{x+ct} g(\tau)d\tau - \frac{k}{2}$ ……..(4)

∴ The solution of the initial -boundary value problem is

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}[f(x+ct) + f(x-ct)] + \tfrac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau \text{ for } x > ct \\ u(x,t) &= \tfrac{1}{2}[f(x+ct) - f(ct-x)] + \tfrac{1}{2c}\int_{ct-x}^{x+ct} g(\tau)d\tau \text{ for } x < ct. \end{aligned} \quad ……… (5)$$

Here f must be twice continuously differentiable and g must be continuously differentiable, $f(0) = f''(0) = g(0) = 0.$

Example 2:

Determine the solution of the initial-boundary value problem

$$\begin{aligned} &u_{tt} = 4u_{xx}, && x > 0, t > 0. \\ &u(x,0) = |\sin x|, && x > 0. \\ &u_t(x,0) = 0, && x \ge 0. \\ &u(0,t) = 0, && t \ge 0. \end{aligned}$$

Solution:

$$u(x,t) = \phi(x+2t) + \psi(x-2t) \longrightarrow (*)$$
$$u(x,0) = \phi(x) + \psi(x) = |\text{ sen } x) \longrightarrow (1).$$
$$\phi(x) + \psi(x) = \frac{1}{2}(0)$$
$$= 0$$

$\Rightarrow \phi(x) - \psi(x) = 0$ .... (2)

(1)+(2), $\Rightarrow 2\phi(x) = |\sin x|$ ....(3)

(1)-2) $\Rightarrow 2\psi(x) = |\sin x|$ ....(4)

For $x > 2t$,

$$(*) \Rightarrow u(x,t) = \frac{1}{2}|\sin(x+2t)| + \frac{1}{2}|\sin(x-2t)|.$$
$$= \frac{1}{2}[|\sin(x+2t)| + |\sin(x-2t)|]$$
$$\frac{1}{2}\sin 2t + \sin(-2t) =$$

For $x < 2t$, $[\psi(x) = -\phi(x)]$

$$\psi(x-2t) = -\phi(2t-x).$$
$$= -\frac{1}{2}[\sin 2t - x].$$

$(*) \Rightarrow u(x,t) = \frac{1}{2}[|\sin(x+2t)| + (|\sin(2t-x)|)].$

$$= \frac{1}{2}[|\sin(x+2t)| - |\sin(2t-x)|].$$

B) Semi-infinite string with a free End:

consider a semi- infinite string with a free end at $x = 0$.

$u_{tt} = c^2 u_{xx}$, $\quad 0 < x < \infty, t > 0$
$u(x,0) = f(x)$, $\quad 0 \leq x < \infty$.
$u_t(x,0) = g(x)$, $\quad 0 \leq x < \infty$.
$u_x(0,t) = 0$, $\quad 0 \leq t < \infty$.

Solution:

For $x < ct$,

$$u(x,t) = \phi(\text{x} + ct) + \psi(x-ct).$$
$$u_x(x,t) = \phi'(x+ct) + \psi'(x-ct).$$
$$u_x(0,t) = \phi'(ct) + \psi'(-ct) = 0.$$

Integrating, $\int \phi'(ct) + \psi'(-ct)dt = 0$.

$\frac{1}{c}[\phi^*(ct) - \psi(-ct)] = k, x$ i a constant.

$\Rightarrow \phi(ct) - \psi(-ct) = k \quad \ldots \ldots (1)$

Let $\alpha = -ct$, in ϕ.

$\phi(-\alpha) - \psi(\alpha) = \mathrm{k}$

$-\psi(\alpha) = -\phi(-\alpha) + k.$

$\psi(\alpha) = \phi(-\alpha) - k.$

Replacing α by $x - ct$,

$$\psi(x - ct) = \phi(ct - x) - k.$$

$$\psi(x - ct) = \frac{1}{2}f(ct - x) + \frac{1}{2c}\int_0^{ct-x} g(\tau)d\tau - \frac{k}{2}$$

The solution of the initial boundary value problem,

$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$ for $x > ct$

$u(x,t) = \frac{1}{2}[f(x+ct) + f(ct-x)] + \frac{1}{2c}\left[\int_0^{x+ct} g(\tau)d\tau + \int_0^{ct-x} g(t)\right.$. for $x < ct$.

Here f must be twice continuously differentiable and g must be continuously differentiable,

$f'(0) = g'(0) = 0.$

Example 3:

Find the solution of the initial boundary value prole.

$u(x,0) = \cos\frac{\pi x}{2}, \quad 0 \le x \le \infty$

$u_t(x,0) = 0, \quad 0 \le x < \infty.$

$u_x(0,t) = 0, \quad t \ge 0.$

Solution:

Given $u_{tt} = u_{x1}$, Here c^2=1 , c=1.

For $x > t$,

$\therefore c = 1.$

We know that

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)dz.$$

$$= \frac{1}{2}\left[\cos\frac{\pi}{2}(x+t) + \cos\frac{\pi}{2}(x-t)\right] \; [\because c = 1, g(z) = 0]$$

$$= \frac{1}{2}\left[x\cos\frac{\pi}{2}x\left[\frac{x+y+x-t}{2}\right] * \cos\frac{\pi}{2} \times \left(\frac{xx+t-x+t}{2}\right)\right].$$

$$= \frac{1}{2}\cos\frac{\pi}{2} \times \cos\frac{\pi}{2}t.$$

$u(x,t)$ for $x < t$,

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(ct-x)] + \frac{1}{2c}\left[\int_0^{x+ct} g(\tau)dr + \int_0^{xt-x} g(\tau)d\tau\right]$$

$$= \frac{1}{2}\left[\cos\frac{\pi}{2}(x+ct) + \cos\frac{\pi}{2}(ct-x)\right] + 0.$$

$$= \frac{1}{2}\left[2\cos\frac{\pi}{2}(x+ct+ct-x) \times \cos\frac{\pi}{2}(x+ct-ct+x)\right]$$

$$= \cos\frac{\pi}{2}ct\cos\frac{\pi}{2}x$$

$$= \cos\frac{\pi}{2}t\cos\frac{\pi}{2}x \; [\because c = 1].$$

2.5. Non- homogeneous Boundary conditions:

$$u_{tt} = c^2uxx,\ x > 0, t > 0.$$

$$u(x,0) = f(x),\ x \geq 0.$$

$$u_t(x,0) = g(x),\ x \geq 0.$$

$$u(0,t) = p(t),\ t \geq 0.$$

$$W \cdot K \cdot T\ u(x,t) = \phi(x+ct) + \psi(x-ct).$$

$$u(0,t) = \phi(ct) + \psi(-ct) = p(t).$$

put $\alpha = -ct$.

$$\phi(-\alpha) + \psi(\alpha) = p\left(\frac{-\alpha}{c}\right).$$

$$\Rightarrow\ \psi(\alpha) = p\left(-\frac{\alpha}{c}\right) - \phi(-\alpha)$$

Replacing α by $x - ct$;

$$\psi(x-ct) = P\left(\frac{ct-x}{c}\right) - \phi(ct-x)$$

$$= P\left(t - \frac{x}{c}\right) - \phi(ct-x)$$

For $x < ct$,

$$u(x,t) = \varphi(x+ct) + \psi(x-ct).$$

$$= \quad \varphi\left(t-\frac{x}{c}\right) + \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\int g(\tau)d\tau.$$

$$= p\left(t-\frac{x}{c}\right) + \phi(x+ct) - \psi(xt-x).$$

where $\phi(\varepsilon) = \frac{1}{2}f(\varepsilon) + \frac{1}{2c}\int_0^{\varepsilon} g(\tau)d\tau$.

$$\psi(\eta) = \frac{1}{2} + (\eta) + \frac{1}{2c}\int_0^{\eta} g(\tau)d\tau.$$

For $x > ct$,

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f\binom{x-ct}{t-x}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)dr$$

provided f must be twice diff contivicuouly f must be continuously differentiable, P must be twice continuously diff in t,

$p(0) = f(0), p'(0) = g(0), p''(0) = c^2 f''(0).$

case (ii):

let us consider the initial boundary value problem.

$$u_{tt} = c^2 u_{xx},\ x > 0, t > 0.$$
$$u(x,0) = f(x),\ x \geq 0.$$
$$u_t(x,0) = g(x),\ x \geq 0.$$
$$u_x(0,t) = q(t),\ .t \geq 0.$$

We know that,

$$u(x,t) = \phi(x+ct) - \psi(x-ct).$$
$$u_x(x,t) = \phi'(x+ct) + (x-ct).$$

Apply initial boundary conditions,

$u_x(0,t) = \phi'(ct) + \psi'(-ct) = q(t).$

Integrating, $\frac{1}{c}\phi(ct) - \frac{1}{c}\psi(-ct) = \int_0^t q(\tau)d\tau + k$

$$\phi(ct) - \psi(-ct) = c\int_0^t q(\tau)d\tau + k.$$

if we let $\alpha = -ct$, then .

$$\phi(-\alpha) - \psi(\alpha) = c\int_0^t q(\tau)d\tau + k.$$

$$\psi(\alpha) = \phi(-\alpha) - c\int_0^{-\alpha/c} q(\tau)d\tau - k$$

Replace a by x-ct.

$$\psi(x-ct) = p(ct-x) - c\int_0^{t-\frac{x}{c}} q(\tau)d\tau - k.$$

$$u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(\tau)d\tau + \frac{k}{2} + \frac{1}{2}t(ct-x)$$
$$\frac{1}{2c}\int_0^{ct-x} g(\tau)d\tau + \frac{k}{2} - c\int_0^{t-x/c} q(\tau)d\tau - k.$$

For $x < ct$,

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(ct-x)] + \frac{1}{2c}\left[\int_0^{x+ct} g(\tau)d\tau + \int_0^{ct-x} g(\tau)d\tau\right] - c\int_0^{t-x/c} \mathrm{q}(\tau)d\tau.$$

provided f is twice continuously differential y, and g must be continuously differ.

In addition $f'(0) = q(0), g'(0) = q'(0)$.

Note:

we can use the elastic boundary end in the same manner.

$u_x(0,t) + hu(0,t) = 0,\ h =$ constant.

2.6. Finite string with fixed Ends:

$$\begin{aligned}
&u_t = c^2 u_{xx}, && 0 < x < 1, t > 0\\
&u(x,0) = f(x), && 0 \le x \le 1.\\
&u_t(x,0) = g(x), && 0 \le x \le 1.\\
&u(0,t) = 0 && t \ge 0.\\
&u(l,t) = 0. && t \ge 0.
\end{aligned}$$

we know that $u(x,t) = \varphi(x+ct) + \psi(x-ct)$.

Apply the initial ends.

$$\left.\begin{aligned}
u(x,0) &= p(x) + \psi(x) = f(x)\ 0 \leqslant x \leqslant l \to (t)\\
u_t(x,t) &= c\varphi'(x+ct) - c\psi'(x-ct) = g(x)
\end{aligned}\right. \ 0 \le x \le l$$

We know that $\varphi(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\varepsilon} g(\tau)d\tau + \frac{k}{2} \quad 0 \le \xi \le l.$ ……… (A)

$$\varphi(\eta) = 1f(\eta) - 1$$

$$\varphi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)d\tau - \frac{k}{2} \quad 0 \le \eta \le l \quad \ldots\ldots (B)$$

$$\therefore u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau.$$

for $0 \leqslant x + ct \le l$ and $0 \leqslant x - ct \le l$.

The solution is determined by the initial data in the region,

$$0 \le x + ct \le l \Rightarrow -x \le ct \le l - x.$$

$$-\frac{x}{c} \le t \le \frac{l-x}{c}.$$

$$\therefore t \le \frac{x}{c},\ t \le \frac{l-x}{c}, t \ge 0.$$

Applying boundary cads,

$u(0,t) = \varphi(ct) + \psi(-ct) = 0, t \geq 0 \quad \dots\dots\dots (1).$
$u(l,t) = \varphi(l+ct) + \psi(l-ct) = 0, t \geq 0 \dots\dots\dots (2)$

set $\alpha = -ct$,

$(1) \Rightarrow \varphi(-\alpha) + \psi(\alpha) = 0$
$\Rightarrow \psi(\alpha) = -\varphi(-\alpha), \qquad \alpha \leq 0. \dots\dots. (3).$

if we set $\alpha = l + ct$ in (2).

$\varphi(\alpha) + \psi(l - (\alpha - l)) = 0.$
$\varphi(\alpha) = -\psi(2l - \alpha), \alpha \geq l \ \dots\dots. (4).$

Put $\xi = -\eta$ in(1)

$$\varphi(-\eta) = \frac{1}{2} f(-\eta) + \frac{1}{2}\int_0^{-\eta} \left(g(\tau)d\tau + \frac{k}{2}\right),$$

From (4) and (5),

$$\varphi(\pi) = -\frac{1}{2} f(-\eta) - \frac{1}{2c}\int_0^{-\pi} g(\tau)d\tau - \frac{k}{2}, \qquad -1 \leq \eta \leq 0 \quad \dots\dots\dots\dots (6)$$

Rang of φ (m) is extended to $a - 1 \leqslant m < 1$.

Put $a = 5$ in $(4) \Rightarrow \varphi(\xi) = -\psi(2l - \xi)$, $\xi \geq$ l] ....(7)

Put $\eta = 2l - \xi$ in (B)

$$\varphi(2\,l - \xi) = \frac{1}{2f(2\,l-\xi)} - \frac{1}{2c}\int_0^{2l\,-\varepsilon} g(\tau)d\tau - \frac{k}{2}, 0 \leq 2\,l - \xi \leq l \quad \dots\dots\dots\dots (8)$$

Sub in (7)

$$\varphi(\xi) = -\frac{1}{2} f(2l - \xi) + \frac{1}{2c}\int_0^{2\,l-\varepsilon} g(\tau)d\tau + \frac{k}{2},\ l \leq \xi \leq 2l\,,$$

The range of $\varphi(\xi)$ is extended to $0 \leq \varepsilon \leq 2l$. continuing in this manner, $\varphi(\xi)$ Hence the solution is determined $\forall\ 0 \leq x \leq l$. and $l \geq 0$.

In region 1- only direct waves propagate

In regions 2 and 3 - Both direct and reflected waves propagate.

In region 4,5,6,... – several waves propagate along is characteristics reflected from both of the boundaries $x = 0$ and $x = l$.

Example 4:

Determine the sol of $u_{tt} = c^2 u_{xx}\ 0 < x < l,\ t > 0$.

$u(x,0) = \sin\left(\frac{\pi x}{l}\right). \quad 0 \leq x \leq l$
$u_t(x,0) = 0. \qquad 0 \leq x \leq l$
$u(0,t) = 0. \qquad t \geq 0$
$u(l,t) = 0. \qquad t \geq 0.$

Proof:

$$\left[We\ know\ that, \phi(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\xi} g(\tau)d\tau + \frac{k}{2},\ 0 \le \xi \le l\right.$$

$$\left.\psi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)d\tau - \frac{k}{e},\ 0 \le \eta \le l\right]$$

$$\Rightarrow \phi(\xi) = \frac{1}{2}\sin\frac{\pi\xi}{l} + \frac{k}{2},\ 0 \le \xi \le l.$$

$$\Rightarrow \psi(\eta) = \frac{1}{2}\sin\frac{\pi\eta}{l} - \frac{k}{2}\ ,\ 0 \le \eta \le l.$$

$$\left[\psi(\eta) = -\frac{1}{2}f(-\eta) - \frac{1}{2c}\int_0^{-\eta} g(\tau)d\tau - \frac{k}{2},\ -l \le \eta \le 0\right].$$

$$\Rightarrow \psi(\eta) = -\frac{1}{2}\sin\left(-\frac{\pi\eta}{l}\right) - \frac{k}{2}.\ , -l \le \eta \le 0.$$

$$= \frac{1}{2}\sin\left(\frac{\pi\eta}{l}\right) - \frac{k}{2}(\because \sin(-\theta) = -\sin\theta)$$

$$\left[\phi(\xi) = -\frac{1}{2}f(2l-\xi) + \frac{1}{2c}\int_0^{2l-\xi} g(\tau)d\tau + \frac{k}{2},\ l \leqslant \xi \leqslant 2l\right].$$

$$\Rightarrow \phi(\xi) = -\frac{1}{2}\sin_{1}^{\frac{\pi}{2}}(2l-\xi) + \frac{k}{2}\ l \le \xi \le 2l.$$

$$[\phi(\alpha) = \phi(\alpha), \psi(\alpha) = -\phi(-\alpha), \alpha \le 0].$$

$$\therefore \psi^{\eta} = -\left[-\frac{1}{2}\sin\frac{\pi}{l}(2l+\eta) + \frac{k}{2}\right], -2l \le \eta \le -l$$

$$\psi(\alpha) = \frac{1}{2}\sin\frac{\pi}{l}\eta - \frac{k}{2},\ -2l \leqslant \eta \leqslant -l.$$

proceeding in this manner, we determine the solution

$$u(x,t) = \phi(\xi) + \psi(\eta).$$

$$= \frac{1}{2}\sin\frac{\pi}{l}(\xi) + \frac{1}{2}\sin\frac{\pi}{l}(\eta) + \frac{16}{2} - k/2.$$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\sin\frac{\pi}{l}(x+ct) + \sin\frac{\pi}{l}(x-c,t)\right] \qquad \begin{matrix}\xi = x+ct \\ \eta = x-ct.\end{matrix}$$

x in $(0,l)$ and $\forall t > 0$.

2.7. Non-Homogeneous Wave Equation:

The Cauchy problem for the non-homogeneous wave equation.

$$u_{tt} = c^2 u_{xx} + h^*(x,t) \qquad \ldots\ldots\ldots (1)$$

with the initial conditions,

$$u(x,0) = f(x)$$
$$u_t(x,0) = g^*(x)$$

By the coordinate transformation,

$y = ct. \Rightarrow t = y/c \Rightarrow \frac{dt}{dy} = \frac{1}{c}$

Now, consider $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}\frac{dt}{dy}$

$$u_y = (u_t)\frac{1}{c}. \Rightarrow u_t = cu_y \ldots\ldots\ldots (3).$$

i.e.) $\frac{\partial^2 u}{\partial y^2} = u_{tt}\frac{1}{c^2}.$

$$\Rightarrow u_{yy} = \frac{u_{tt}}{c^2}.$$

$\Rightarrow u_{tt} = c^2 u_{yy} \ldots\ldots\ldots (4)$

sub(3) & (4) in (1) & (2).

$\Rightarrow c^2 u_{yy} = c^2 u_{xx} + h^*(x,t)$

$$u_{yy} = u_{xx} + \frac{h^*(x,t)}{c^2} \qquad h(x,y) = -\frac{h^*}{c^2}$$

$$u_{yy} = u_{xx} - h(x,y)$$

U(x,0) =f(x) ……….. (6)

$cu_y = g^*(x)$

$u_y = g^*(x)/$c. where $g(x) = g^*/c$

$u_y(x,0) = g(x) \ldots\ldots\ldots (7)$

Let $P_0(x_0, y_0)$ be a point of the plane, let Q_0 be the point $(x_0, 0)$ on the initial line $y = 0$. Then the characteristics $x \pm y =$ constant of equation (5) are two straight lines drawn through the point P_0 with slopes ± 1. They intersect the x-axis at the points $P_1(x_0 - y_0, 0)$ and $P_2(x_0 + y_0, 0)$. Let the sides of the triangle $P_0P_1P_2$ be designated by B_0, B_1, and. B_2. Let R be the region representing the interior of the triangle and its Boundaries B.

Now Integrating equation (5), we obtain.

$$\iint_R (u_{xx} - u_{yy})dR = \iint_R h(x,y)dR \ldots\ldots\ldots (8)$$

Using, Green's theorem; we obtain

$$\iint_R (u_{xx} - u_{yy})dR = \oint_B (u_x dy + u_y dx) \ldots\ldots\ldots\ldots (9)$$

Since B is composed of B_0, B_1 and B_2,

$\int_{B_0} (u_x dy + u_y dx) = \int_{(x_0-y_0)}^{(x_0+y_0)} u_y dx$ ………. (10) [Here, a & B_0, y=0 dy=0]

$$\int_{B_1} (u_x dy + u_y dx) = +\int_{(x_0,y_0)}^{(x_0+y_0,0)} (u_x dx + u_y dy)$$

$$= +\int_{(x_0,y_0)}^{(x_0+y_0,0)} d(u(x,y)) = -\int_{(x_0+y_0,0)}^{(x_0,y_0)} d(u(x,y)).$$

$$= -[u(x,y)]_{(x_0+y_0,0)}^{(x_0,y_0)}$$
$$= -[u(x_0,y_0) - u(x_0+y_0,0)]$$
$$= u(x_0+y_0,0) - u(x_0,y_0) \rightarrow (11):$$

$$\int_{B_2} (uxdy + uydx) = \int_{B_2} (uxdx + uydy)$$
$$= u(x_0-y_0,0) - u(x_0,y_0) \rightarrow \ (12)$$

Adding (10), (11) & (2).

$$\oint (uxdy + uydx) = -2u(x_0,y_0) + u(x_0-y_0,0) + u(x_0+y_0,0).$$
$$+\int_{x_0-y_0}^{x_0+y_0} uydx$$

$$u(x_0,y_0) = -\frac{1}{2}\oint (u_x dy + u_y dx) + \frac{1}{2}[u(x_0+y_0,0) +$$
$$u(x_0-y_0,0)] + \frac{1}{2}\int_{x=y_0}^{x_0+y_0} a_y dx.$$
$$= \frac{-1}{2}\iint_k h(x,y)dR + \frac{1}{2}[u(x_0+y_0,0) + u(x_0-y_0,0)]$$
$$+\frac{1}{2}\int_{x_0-y_0}^{x_0+y_0} u_y dy \qquad [by\ (9)\&\ (8)]. \quad \ldots\ldots\ldots\ldots\ldots..(14)$$

we Replace x_0 by x & y_0 by y.

(14) ⇒ $\mathrm{u(x,y)} = \frac{1}{2}[f(x+y) + f(x-y)] + \frac{1}{2}\int_{x-\mathrm{y}}^{x+\mathrm{y}} g(\tau)\cdot d\tau - \frac{1}{2}\iint_R h(x,y)dR.$

Now, $y = ct$.

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x=ct}^{xc+ct} g^*(\tau)\cdot d\tau -$$
$$\frac{1}{2}\iint_R h(x,t)dR.$$

Example 5:

Determine the solution of

$u_{xx} - u_{yy} = 1.$
$u(x,0) = \sin x.$
$u_y(x,0) = x.$

Solution:

$$\text{u(x,y)} = \frac{1}{2}[f(x+y)+f(x-y)] + \frac{1}{2}\int_{x-\text{y}}^{x+\text{y}} g(\tau)\cdot d\tau - \frac{1}{2}\iint_R h(x,y)dR.$$

Now the characteristics are $x + y = x_0 + y_0$,

$x - y = x_0 - y_0$,

$$\text{u}(x_0, y_0) = \frac{1}{2}[sin(x_0+y_0) + sin(x_0-y_0)] + \frac{1}{2}\int_{x_0-y_0}^{x_0+y_0}(\tau)\cdot d\tau$$

$$x - y = x_0 - y_0.$$

$$-\frac{1}{2}\int_0^{y_0}\int_{y+x_0-y_0}^{-y+x_0+y_0} dxdy.\frac{1}{2}\int_{x_0-y_0}^{x_0+y_0}\tau d\tau$$

Now dropping the subscript zero,

$$\Rightarrow u(x,y) = \frac{1}{2}[\sin(x+y)+\sin(x-y)] + \frac{1}{2}\int_{x-y}^{(x+y)}\tau d\tau$$

$$-\frac{1}{2}\int_0^{y}\int_{y+x-y}^{-y+x+y} dxdy.$$

$$= \frac{1}{2}[\sin(x+y)+\sin(x-y)] + \frac{1}{2}\left[\frac{\tau^2}{2}\right]_{x-y}^{x+y} - \frac{1}{2}(y)$$

2.8 Riemann method:

The Linear Hyperbolic equation $L[u] = u_{xy} + au_x + bu_y + cu = f(x,y) \to (1)$

denotes the linear operator $a(x,y), b(x,y), c(x,y)$ & $f(x,y)$ are differentiable functions in some domain D^*.

Let $v(x,y)$ be a term having continuous second-order partial derivatives.

$$\text{Let } (vu_x)_y(uv_y)_x = vu_{xy} + u_xv_y - [uv_{yx} + v_yu_x].$$

$$= vu_{xy} + u_xb_y - uv_{yx} - v_yu_x.$$

$$\Rightarrow (vu_x)_y - (u\cdot v_y)_x = vu_{xy} - uv_{xy} \longrightarrow (2)$$

$$\text{Let } (vau_x - u(av)_x = vau_x + av_xu + a_xub - uab_x - ua_xv.$$

$$= vau_x.$$

$$\Rightarrow (vau)_x - u(av)_x = vau_x. \quad \ldots\ldots\ldots (3)$$

$$u_y \Rightarrow (vbu)_y - u(abv)_y = vbu_y$$

Now consider $vL[u] - uM[v] = vu_{xy} + \underbrace{vau_x} + vbu_y + vcu$

$-uv_{xy} + u(av)_x + u(bv)_y - ucv.$

$= vu_{xy} + vau_x + vbu_y + bcu - (vv_{xy})$

$= (vu_x)_y - (uv_y)_x + (vau)_x - u(dv)_x + (vbu)_y - u(abv)_y$

$vcu + u(av)_x + u(bv)_y - bcv$

$\therefore u\,L[u] - um[v] = u_x + u_y \;\; \ldots\ldots..(4)$

where M is the operator represented by.

$M[v] = v_{xy} - (av)_x - (bv)_y + cv$ ……….. (5)

And $u = auv - uv_y \;\; v = buv + vu_x$

$$\left[uL[u] - uM[v] = (vau - uv_y)_x + (vbu + vu_x)_y\right]. \quad \ldots\ldots\ldots\ldots (6)$$

The operator M is called the adjoint operator of L.

Note: If $M = L$, then the operator L is said to be. self -adjoint.

Now applying Greens theorem,

$\iint_D (v_x + v_y)dxdy = \oint_C (udy - vdx) \quad \ldots\ldots\ldots$ (7)

where c is the closed curve bounding the region of integration D which is in D^*.

Let $\wedge$ be a smooth initial curve. We assume that the tangent to A is nowhere parallel to $x\ or\ y$ axes. Let $p(\alpha,\beta)$ be a point at which the solution to the Cauchy problem is sought.. Line PQ parallel to x-axis intersect y the initial curve $\wedge$ at Q and the line PR parallel to the y-ali intersects the curve $\wedge$ at R. Let e be closed contour $PQRP$ bound. since $dy = 0$ on PQ, and $dx = 0$ on PR. combine (4) & (7).

$$\begin{aligned}\iint_D (vL(u) - uM[v])dxdy &= \iint_D (U_x + v_x)dxdy.\\ &= \oint_C udy - vdx \qquad\qquad \ldots\ldots.. (8)\\ &= \int_Q^R udy - vdx + \int_R^P udy - \int_P^Q vdx\end{aligned}$$

consider $\int_P^Q vdx = \int_P^Q bvudx + \int_P^Q vu_x dx.$

$$\left[\begin{array}{l}u = v,\\ du = v_x dx,\ v = u_x dx.\end{array}\right.$$

$$\int_P^Q vdx = \int_P^Q bvudx + [uv]_P^Q - \int_P^Q uv_x dx.$$

$$= [uv]_P^Q + \int_P^Q u(bv - v_x)dx \rightarrow \text{(uv)}.$$

sub (8) in (7)

$$\iint_D (v+(u)-uM(u))dxdy = \int_Q^R udy - vdx + \int_R^P udy - (uv)_R + (uv)_P - \int_P^Q u(bv - v_x)dx.$$

$$\Rightarrow (uv)_P = \left\{(uv)_Q - \int_R^P u(av - v_y)dy + \int_P^Q u(bv - v_x)dx - \int_Q^R vdy - vdx + \iint_D (v + (u) - uM(v))dxdy \right. \quad \ldots\ldots.. (9)$$

Suppose we can choose the fun $v(x, y; \alpha, \beta)$ be the son of the adjoint equation

$M(v) = 0 \quad \ldots\ldots\ldots (10)$

satisfying theorem,

$v_x = bv$ when $y = \beta$.

$v_y = av$ when $x = \alpha$.

$\because v = 1$ when $x = \alpha$ and $y = \beta\} \quad \ldots\ldots\ldots(11)$

The function $v(x, y, \alpha, \beta)$ is called the Riemann function. since $\alpha[u] = f$

Equation (9) reduces,

$$[u]_p = [uv]_Q + \int_Q^R (auv - uv_y)dy - (buv + vu_x)dx + \iint_D vfdxdy$$

$$= [uv]_Q - \int_Q^R uv(ady - bdx) + \int_Q^R uv_ydy + vu_xdx + $$

$$= \iint_D fdxdy \quad \ldots\ldots\ldots. (12)$$

This gives us the value of u at the point p when u and u_x are prescribed along the curve $\wedge$. when $uf\ u_y$ are prescribed, the identity.

$$[uv]_R - [uv]_Q = \int_Q^R [(uv)_x dx + (uv_v)_y dy].$$

$$[uv]_Q = [uv]_R - \int_Q^R (uv)_x dx + (uv)_y dy \quad \ldots\ldots\ldots (13).$$

sab (13) in (9).

$$[u]_P = [uv]_R - \int_Q^R (ux)_x dx + (uv)_y dy - \int_Q^R uv(ady - bdx).$$

$$+ \int_0^R vu_x dx + uv_y dy + \iint_D vfdxdy \quad \ldots\ldots\ldots (14)$$

Add equation (12) and (14),

$$2[u]_P = [uv]_Q + [uv]_R - 2\int_Q^R uv(ady - bdx) + 2\iint_D vf dxdy$$
$$- \int_Q^R u[v_x dx - v_y dy] + \int_Q^R v[u_x dx - u_y dy]$$
$$[u]_P = \frac{1}{2}[[uv]_Q + [uv]_R] - \int_Q^R uv(ady - bdx) + \iint_D vf dxdy$$
$$-\frac{1}{2}\int_Q^R u[v_x dx - v_y dy] + \frac{1}{2}\int_Q^R v[u_x dx - u_y dy]$$

where

$$G(x,t,\tau) = \left\{-2\sqrt{k}ctf(\tau)J_0\left[\sqrt{4k[(\tau-x)^2 - c^2t^2]}\right]\right\}/\sqrt{(\tau-x)^2 - c^2t^2}$$
$$+c^{-1}g(\tau)J_0\left\{\sqrt{4k[(\tau-x)^2 - c^2t^2]}\right\}$$

If we set $k = 0$, we arrive at the d'Alembert formula for the wave equation

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$$

2.9 Goursat Problem:

The Goursat problem is that of finding the solution of a linear hyperbolic equation

$u_{xy} = a_1(x,y)u_x + a_2(x,y)u_y + a_3(x,y)u + h(x,y)$ .... (1)

satisfying the prescribed conditions $u(x,y) = f(x)$ ....(2)

on a characteristic, say, $y = 0$, and $u(x,y) = g(x)$ ....(3)

on a monotonic increasing curve $y = y(x)$ which, for simplicity, is assumed to intersect the characteristic at the origin.

The solution in the region between the x-axis and the monotonic curve in the first quadrant can be determined by the method of successive approximations.

Example 1:

Determine the solution of the Goursat problem

$u_{tt} = c^2 u_{xx}$ ... ... (4)
$u(x,t) = f(x)$ on $x - ct = 0$ ... ... (5)
$u(x,t) = g(x)$ on $t = t(x)$ ... ... (6)

where $f(0) = g(0)$.

The general solution of the wave equation is

$u(x,t) = \phi(x+ct) + \psi(x-ct)$

Applying the prescribed conditions, we obtain

$f(x) = \phi(2x) + \psi(0)$ ……………..(7)

$g(x) = \phi\big(x + ct(x)\big) + \psi\big(x - ct(x)\big)$ …………… (8)

It is evident that

$f(0) = \phi(0) + \psi(0) = g(0)$

Now, if $s = x - ct(x)$, the inverse of it is $x = \alpha(s)$. Thus, Eq. (4.9.8) may be written as

$g\big(\alpha(s)\big) = \phi\big(x + ct(x)\big) + \psi(s)$ ……………… (9)

Replacing x by $(x + ct(x))/2$ in Equation (7), we obtain

$f\left(\frac{x+ct(x)}{2}\right) = \phi(x + ct(x)) + \psi(0)$ …………… (10)

Thus, using (10), (9) becomes

$$\psi(s) = g(\alpha(s)) - f\left(\frac{\alpha(s) + ct(\alpha(s))}{2}\right) + \psi(0)$$

Replacing s by $x - ct$, we have

$$\psi(x - ct) = g(\alpha(x - ct)) - f\left(\frac{\alpha(x - ct) + ct(\alpha(x - ct))}{2}\right) + \psi(0)$$

Hence, the solution is given by

$u(x,t) = f\left(\frac{x+ct}{2}\right) - f\left(\frac{\alpha(x-ct)+ct(\alpha(x-ct))}{2}\right) + g(\alpha(x - ct))$ ………… (11)

Let us consider a special case when the curve $t = t(x)$ is a straight line represented by $t - kx = 0$ with a constant $k > 0$. Then $s = x - ckx$ and hence $x = s/(1 - ck)$. Using these values in (4.9.11), we obtain

$u(x,t) = f\left(\frac{x+ct}{2}\right) - f\left(\frac{(1+ck)(x-ct)}{2(1-ck)}\right) + g\left(\frac{x-ct}{1-ck}\right)$ ………… (12)

When the values of u are prescribed on both the characteristics, the problem of finding u of a linear hyperbolic equation is called a characteristic initial value problem. This is the degenerate case of the Goursat problem.

Consider the characteristic initial value problem

$u_{xy} = h(x,y)$ ………….. (13)

$u(x,0) = f(x)$ ………….. (14)

$u(0,y) = g(y)$ ………….. (15)

where f and g are continuously differentiable and $f(0) = g(0)$.

Integrating Equation (13), we obtain

$$u(x,y) = \int_0^x \int_0^y h(\xi,\eta)d\eta d\xi + \phi(x) + \psi(y) \quad \ldots\ldots\ldots\ldots (16)$$

where ϕ and ψ are arbitrary functions. Applying the prescribed conditions (14) and (15), we have

$u(x,0) = \phi(x) + \psi(0) = f(x) \quad \ldots\ldots\ldots\ldots (17)$

$u(0,y) = \phi(0) + \psi(y) = g(y) \quad \ldots\ldots\ldots\ldots (18)$

Thus $\phi(x) + \psi(y) = f(x) + g(y) - \phi(0) - \psi(0) \quad \ldots\ldots\ldots\ldots(19)$

But from (17), we have $\phi(0) + \psi(0) = f(0) \quad \ldots\ldots\ldots\ldots(20)$

Hence, from (16), (19), and (20), we obtain

$$u(x,y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y h(\xi,\eta)d\eta d\xi \quad \ldots\ldots\ldots\ldots(21)$$

Example 2:

Determine the solution of the characteristic initial value problem

$$u_{tt} = c^2 u_{xx}$$

$u(x,t) = f(x)$ on $x + ct = 0$

$u(x,t) = g(x)$ on $x - ct = 0$

where $f(0) = g(0)$.

Here it is not necessary to reduce the given equation into canonical form. The general solution of the wave equation is

$u(x,t) = \phi(x + ct) + \psi(x - ct)$

The characteristics are

$x + ct = 0$

$x - ct = 0$

Applying the prescribed conditions, we have

$u(x,t) = \phi(2x) + \psi(0) = f(x)$ on $x + ct = 0 \quad \ldots\ldots\ldots\ldots (22)$

$u(x,t) = \phi(0) + \psi(2x) = g(x)$ on $x - ct = 0 \quad \ldots\ldots\ldots\ldots (23)$

We observe that these equations are compatible, since $f(0) = g(0)$.

Now, replacing x by $(x + ct)/2$ in Equation(22) and replacing x by $(x - ct)/2$ in Equation(23), we have

$$\phi(x + ct) = f\left(\frac{x + ct}{2}\right) - \psi(0)$$

$$\psi(x - ct) = g\left(\frac{x - ct}{2}\right) - \phi(0)$$

Hence the solution is given by

$$u(x,t) = f\left(\frac{x+ct}{2}\right) + g\left(\frac{x-ct}{2}\right) - f(0) \quad \ldots\ldots\ldots\ldots (24)$$

We note that this solution can be obtained by substituting $k = -1/c$ into (12)

Example 3:

Find the solution of the characteristic initial value problem

$y^3u_{xx} - yu_{yy} + u_y = 0$ ……….. (25)

$$u(x, y) = f(x) \text{ on } x + \frac{y^2}{2} = 4 \text{ for } 2 \leqslant x \leqslant 4$$

$$u(x, y) = g(x) \text{ on } x - \frac{y^2}{2} = 0 \text{ for } 0 \leqslant x \leqslant 2$$

with $f(2) = g(2)$.

Since the equation is hyperbolic except for $y = 0$, we reduce it to the canonical form

$u_{\xi\eta} = 0$

where $\xi = x + y^2/2$ and $\eta = x - y^2/2$. Thus, the general solution is

$$u(x, y) = \phi\left(x + \frac{y^2}{2}\right) + \psi\left(x - \frac{y^2}{2}\right) \quad \ldots\ldots\ldots\ldots (26)$$

Applying the prescribed conditions, we have

$f(x) = \phi(4) + \psi(2x - 4)$ … … … … … … (27)

$g(x) = \phi(2x) + \psi(0)$ … … … … … (28)

Now, if we replace $(2x - 4)$ by $(x - y^2/2)$ in (27) and $(2x)$ by $(x + y^2/2)$ in (28), we obtain

$$\psi\left(x - \frac{y^2}{2}\right) = f\left(\frac{x}{2} - \frac{y^2}{4} + 2\right) - \phi(4)$$

$$\phi\left(x + \frac{y^2}{2}\right) = g\left(\frac{x}{2} + \frac{y^2}{4}\right) - \psi(0)$$

Thus

$$u(x, y) = f\left(\frac{x}{2} - \frac{y^2}{4} + 2\right) + g\left(\frac{x}{2} + \frac{y^2}{4}\right) - \phi(4) - \psi(0)$$

But from (27) and (28), we see that

$f(2) = \phi(4) + \psi(0) = g(2)$

Hence

$$u(x, y) = f\left(\frac{x}{2} - \frac{y^2}{4} + 2\right) + g\left(\frac{x}{2} + \frac{y^2}{4}\right) - f(2)$$

2.10. Spherical Wave Equation:

In spherical polar coordinates (r, θ, ϕ), the wave equation form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \ldots\ldots\ldots\ldots (1)$$

Solutions of this equation are called spherical symmetric waves if u depends on r and t. Thus the solution $u = u(r,t)$ which satisfies the wave equation with spherical symmetry in three dimensional space is $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ …………(2)

Introducing a new dependent variable $U = ru(r,t)$, this equation reduces to a simple form

$U_{tt} = c^2 U_{rr}$ ………… (3)

This is identical with the one dimensional wave equation (4.3.1) and has the general solution in the form $U(r,t) = \phi(r + ct) + \psi(r - ct)$ ……………. (4)

(or) $u(r,t) = \frac{1}{r}[\phi(r + ct) + \psi(r - ct)]$ …………….. (5)

This solution consists of two progressive spherical waves traveling with constant velocity c. The terms involving ϕ and ψ represent the incoming waves to the origin and the outgoing waves from the origin respectively.

Physically, the solution for only outgoing waves generated by a source is of most interest, and has the form $u(r,t) = \frac{1}{r}\psi(r - ct)$ …………(6)

where the explicit form of ψ is to be determined from the properties of the source.

In the context of fluid flows, u represents the velocity potential so that the limiting total flux through a sphere of center at the origin and radius r is

$Q(t) = \lim_{r\to 0} 4\pi r^2 u_r = -4\pi\psi(-ct)$ ………… (7)

In physical terms, we say that there is a simple (or monopole) point source of strength $Q(t)$ located at the origin. Thus the solution (6) can be expressed in terms of Q as

$u(r,t) = -\frac{1}{4\pi r} Q\left(t - \frac{r}{c}\right)$ ………….. (8)

This represents the velocity potential of the point source, and u_r is called the radial velocity. In fluid flows, the difference between the pressure at any time t and the equilibrium value is given by $p - p_0 = \rho u_t = -\frac{\rho}{4\pi r}\dot{Q}\left(t - \frac{r}{c}\right)$ ……………… (9)

where ρ is the density of the fluid. $u(r,0) = f(r)$, $u_t(r,0) = g(r)$, $r \geqslant 0$ ……………. (10)

where f and g are continuously differentiable, is given by

$$u(r,t)=\frac{1}{2r}\left[(r+ct)f(r+ct)+(r-ct)f(r-ct)+\frac{1}{c}\int_{r-ct}^{r+ct}\tau g(\tau)d\tau\right]\ldots\ldots\ldots.(11)$$

provided $r \geqslant ct$. However, when $r < ct$ this solution fails because f and g are not defined for $r < 0$. This initial data at $t = 0, r \geqslant 0$ determine the solution $u(r,t)$ only up to the characteristic $r = ct$ in the $r - t$ plane. To find u for $r < ct$, we require u to be finite at $r = 0$ for all $t \geqslant 0$, that is, $U = 0$ at $r = 0$. Thus the solution for U is

$$U(r,t)=\frac{1}{2}\left[(r+ct)f(r+ct)+(r-ct)f(r-ct)+\frac{1}{c}\int_{r-ct}^{r+ct}\tau g(\tau)d\tau\right]\ldots\ldots\ldots\ldots(12)$$

provided $r \geqslant ct \geqslant 0$, and $U(r,t) = \frac{1}{2}[\phi(ct + r) + \psi(ct - r)]$, $ct \geqslant r \geqslant 0$ …………(13)

where $\phi(ct) + \psi(ct) = 0$ for $ct \geqslant 0$ …………..(14)

In view of the fact that $U_r + \frac{1}{c}U_t$ is constant on each characteristic $r + ct =$ constant, it turns out that

$$\phi'(ct+r)=(r+ct)f'(r+ct)+f(r+ct)+\frac{1}{c}(r+ct)g(r+ct)$$

Or $\phi'(ct) = ctf'(ct) + f(ct) + tg(ct)$

Integration gives

$$\phi(t)=tf(t)+\frac{1}{c}\int_{0}^{t}\tau g(\tau)d\tau+\phi(0)$$

so that $\psi(t) = -tf(t) - \frac{1}{c}\int_0^t \tau g(\tau)d\tau - \phi(0)$

Substituting these values into (13) and using $U(r,t) = ru(r,t)$, we obtain, for $ct > r$,

$$u(r,t)=\frac{1}{2r}\left[(ct+r)f(ct+r)-(ct-r)f(ct-r)+\frac{1}{c}\int_{ct-r}^{ct+r}\tau g(\tau)d\tau\right]\ldots\ldots\ldots\ldots(15)$$

2.11. Cylindrical Wave Equation:

In cylindrical polar coordinates (R, θ, z), the wave equation

$u_{RR} + \frac{1}{R}u_R + \frac{1}{R^2}u_{\theta\theta} + u_{zz} = \frac{1}{c^2}u_{tt}$ ………… (1)

If u depends only on R and t, this equation becomes $u_{RR} + \frac{1}{R}u_R = \frac{1}{c^2}u_{tt}$ …………(2)

Solutions of (2) are called cylindrical waves.

We assume that sources of constant strength $Q(t)$ per unit length are distributed uniformly on the z-axis. The solution for the cylindrical waves produced by the line source is given by the total disturbance

$$u(R,t) = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1}{r}Q\left(t-\frac{r}{c}\right)dz = -\frac{1}{2\pi}\int_{0}^{\infty}\frac{1}{r}Q\left(t-\frac{r}{c}\right)dz \dots\dots\dots\dots (3)$$

where R is the distance from the z-axis so that $R^2 = (r^2 - z^2)$.

Substitution of $z = R\sinh\xi$ and $r = R\cosh\xi$ in (3) gives

$$u(R,t) = -\frac{1}{2\pi}\int_0^{\infty} Q\left(t-\frac{R}{c}\cosh\xi\right)d\xi \quad \dots\dots\dots\dots\dots (4)$$

This is usually considered as the cylindrical wave function due to a source of strength $Q(t)$ at $R = 0$. It follows from (4) that

$$u_{tt} = -\frac{1}{2\pi}\int_0^{\infty} Q''\left(t-\frac{R}{c}\cosh\xi\right)d\xi \quad \dots\dots\dots (5)$$

$$u_R = \frac{1}{2\pi c}\int_0^{\infty} \cosh\xi\, Q'\left(t-\frac{R}{c}\cosh\xi\right)d\xi \quad \dots\dots\dots\dots (6)$$

$$u_{RR} = -\frac{1}{2\pi c^2}\int_0^{\infty} \cosh^2\xi\, Q''\left(t-\frac{R}{c}cosh\,\xi\right)d\xi \quad \dots\dots\dots\dots\dots (7)$$

which give

$$c^2\left(u_{RR}+\frac{1}{R}u_R\right)-u_{tt} = \frac{1}{2\pi}\int_0^{\infty}\frac{d}{d\xi}\left[\frac{c}{R}Q'\left(t-\frac{R}{c}\cosh\xi\right)\sinh\xi\right]d\xi$$

$$= \lim_{\xi\to\infty}\left[\frac{c}{2\pi R}Q'\left(t-\frac{R}{c}\cosh\xi\right)\sinh\xi\right] = 0$$

provided the differentiation under the sign of integration is justified and the above limit is zero. This means that $u(R,t)$ satisfies the cylindrical wave equation (2).

In order to find the asymptotic behavior of the solution as $R \to 0$, we substitute $\cosh\xi = \frac{c(t-\zeta)}{R}$ into (4) and (6) to obtain

$$u = -\frac{1}{2\pi}\int_{-\infty}^{t-R/c}\frac{Q(\zeta)d\zeta}{\left[(t-\zeta)^2-\frac{R^2}{c^2}\right]^{\frac{1}{2}}} \quad \dots\dots\dots\dots (8)$$

$$u_R = \frac{1}{2\pi}\int_{-\infty}^{t-R/c}\left(\frac{t-\zeta}{R}\right)\frac{Q'(\zeta)d\zeta}{\left[(t-\zeta)^2-\frac{R^2}{c^2}\right]^{\frac{1}{2}}} \quad \dots\dots\dots\dots (9)$$

which, in the limit $R \to 0$, give $u_R \sim \frac{1}{2\pi R}\int_{-\infty}^{t} Q'(\zeta)d\zeta = \frac{1}{2\pi R}Q(t)$ …………..(10)

This leads to the result $\lim_{R\to 0} 2\pi R u_R = Q(t)$ ………….(11)

or $u(R,t) \sim \frac{1}{2\pi}Q(t)\log R \;\; as \;\; R$ … … … … … (12)

We next investigate the nature of the cylindrical wave solution near the wave front ($R = ct$) and in the far field ($R \to \infty$). We assume $Q(t) = 0$ for $t < 0$ so that the lower limit of integration in (8) may be taken to be zero, and the solution is non-zero for $\tau = t - \frac{R}{c} > 0$ where τ is the time passed after the arrival of the wave front. Consequently, (8) becomes

$$u(R,t) = -\frac{1}{2\pi}\int_0^{\tau} \frac{Q(\zeta)d\zeta}{\left[(\tau-\zeta)\left(\tau-\zeta+\frac{2R}{c}\right)\right]^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots. (13)$$

Since $0 < \zeta < \tau, \frac{2R}{c} > \frac{R}{c} > \tau > \tau - \zeta > 0$, so that the second factor under the radical is approximately equal to $\frac{2R}{c}$ when $R \gg c\tau$, and hence

$$u(R,t) \sim -\frac{1}{2\pi}\left(\frac{c}{2R}\right)^{\frac{1}{2}} \int_0^{\tau} \frac{Q(\zeta)d\zeta}{(\tau-\zeta)^{\frac{1}{2}}} = -\left(\frac{c}{2R}\right)^{\frac{1}{2}} q(\tau)$$

Where $q(\tau) = \frac{1}{2\pi}\int_0^{\tau} \frac{Q(\zeta)d\zeta}{\sqrt{\tau-\zeta}} \quad \ldots\ldots\ldots\ldots (15)$

Evidently, the amplitude involved in the solution (14) decays like $R^{-\frac{1}{2}}$ for large $R(R \to \infty)$.

Example 1:

Determine the asymptotic form of the solution (4) for a harmonically oscillating source of frequency ω.

We take the source in the form $Q(t) = q_0 \exp[-i(\omega + i\varepsilon)t]$ where ε is positive and small so that $Q(t) \to 0$ as $t \to -\infty$. The small imaginary part ε of ω will make insignificant contributions to the solution at finite times as $\varepsilon \to 0$. Thus the solution (4) becomes

$$u(R,t) = -\frac{q_0}{2\pi} e^{-i\omega t} \int_0^{\infty} \exp\left[\frac{i\omega R}{c}\cosh\xi\right] d\xi$$

where $H_0^{(1)}(z)$ is the Hankel function given by

$$H_0^{(1)}(z) = \frac{2}{\pi i}\int_0^{\infty} \exp(iz\cosh\xi)d\xi \quad \ldots\ldots\ldots\ldots (17)$$

In view of the asymptotic expansion of $H_0^{(1)}(z)$ in the form

$$H_0^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \exp\left[i\left(z - \frac{\pi}{4}\right)\right],\ z \to \infty \quad \ldots\ldots\ldots\ldots..(18)$$

the asymptotic solution for $u(R,t)$ in the limit $\frac{\omega R}{c} \to \infty$ is

$$u(R,t) \sim -\frac{iq_0}{4}\left(\frac{2c}{\pi\omega R}\right)^{\frac{1}{2}} \exp\left[-i\left\{\omega t - \frac{\omega R}{c} - \frac{\pi}{4}\right\}\right]$$

This represents the cylindrical wave propagating with constant velocity c. The amplitude of the wave decays like $R^{-\frac{1}{2}}$ as $R \to \infty$.

Example 2:

For a supersonic flow $(M > 1)$ past a solid body of revolution, the perturbation potential Φ satisfies the cylindrical wave equation $\Phi_{RR} + \frac{1}{R}\Phi_R = N^2\Phi_{xx}$, $N^2 = M^2 - 1$

where R is the distance from the path of the moving body and x is the distance from the nose of the body.

$\Phi_{yy} + \Phi_{zz} = N^2\Phi_{xx}$

This represents a two-dimensional wave equation with $x \leftrightarrow t$ and $N^2 \leftrightarrow \frac{1}{c^2}$. For a body of revolution with $(y, z) \leftrightarrow (R, \theta), \frac{\partial}{\partial\theta} \equiv 0$, the above equation becomes

$$\Phi_{RR} + \frac{1}{R}\Phi_R = N^2\Phi_{xx}$$

UNIT III

Method of separation of variables: Separation of variable- Vibrating string problem – Existence and uniqueness of solution of vibrating string problem - Heat conduction problem – Existence and uniqueness of solution of heat conduction problem – Laplace and beam equations

Chapter 3: Sections 3.1 to 3.6

Method of Separation of variables

3.1. Separation of variables:

consider the second - order homogeneous equation.

$$a^*u_{x^*x^*} + b^*u_{x^*y^*} + c^*u_{y^*y^*} + d^*u_{x^*} + e^*u_{y^*} + f^*u = 0. \quad \ldots\ldots\ldots.. (1)$$

where a^*, b^*, c^*, d^*, e^* and f^* are functions of $x^*\&y^*$.

We know that the transformation.

$$x = x(x^*, y^*)$$
$$y = y(x^*, y^*) \;\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
$$\text{Jacobian } = \frac{\partial(x,y)}{\partial(x^*,y^*)} \neq 0$$

So we can transform (1) in to canonical form.

$$a(x,y)u_{xx} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = 0. \quad \ldots\ldots\ldots\ldots.. (3)$$

mere $B = 0, -4ac = 0.$

i) $a = -c$ is hyperbolic.

ii) $a = 0$ or $c = 0$ is parabolic.

iii) $a = c$ is elliptic:

we assume the solution in the form.

$$u(x,y) = x(x)y(y) \neq 0 \quad \ldots\ldots\ldots\ldots (4)$$

Where x & y are functions of x and y and are twice continuous differentiable.

sub (4) in (3),

$$ax''y + cxy'' + dx'x + exy + fxy = 0. \quad \ldots\ldots\ldots. (5)$$

Let the function $p(x,y)$

we divide equation (5) by $p(x,y)$

$$\Rightarrow a_1(x)x''y + b_1(y)xy'' + a_2(x)x'y + b_2(y)xy' + [a_3(x) + b_3(y)]xy = 0 \ldots\ldots\ldots\ldots.. (6)$$

(6) ÷ by xy

$$\Rightarrow\ a_1\frac{x''}{x}+a_2\frac{x'}{x}+a_3\Big] = -\left[b_1\frac{y''}{y'}+b_2\frac{y'}{y'}+b_3\right] \ldots\ldots\ldots(7)$$

Integrating w.r.to x,

$$\frac{d}{dx}\left[a_1\frac{x''}{x}+a_2\frac{x'}{x}+a_3\right] = 0 \ldots\ldots\ldots(8)$$

Integrating $a_1\frac{x''}{x}+a_2\frac{x'}{x}+a_3 = \lambda\ \ldots\ldots\ldots(9)$

sub in equation (9) in (7).

$$b_1\frac{y''}{y}+b_2\frac{y'}{y}+b_3 = -\lambda \quad \ldots\ldots\ldots\ (10)$$

So, (9) & (10) becomes.

And $a_1x''+a_2x'+(a_3-x)x=0$

$b_1y''+b_2y'+(b_3+\lambda)y=0$ ………….. (11)

Thus $u(x,y)$ is a solution of (3) if $x(x)\&y(y)$ are the solutions of the ordinary diff equation (11) & (12) respectively.

Case (i):

If the coefficients of (1) are constants.

consider the second order equation,

$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0\ \ldots\ldots\ldots(1)$

where A,B,C,D,E & F are constants which are not all zero.

Assume a sold of the form

$u(x,y)=x(x)y(y)\neq 0.$

sub this in (1),

$$Ax''y+Bx'y'+Cxy''+Dx'y+Exy'+Fxy=0\to 0$$
$$(\div Axy)$$
$$\frac{x''}{x}+\frac{Bx'y'}{Ax(y)}+\frac{Cy''}{Ay}+\frac{Dx'}{Ax'}+\frac{Ey'}{Ay'}+\frac{Fxy}{A}=0, A\neq 0. \quad \ldots\ldots\ldots(3)$$

Diff w.r.to x, $\left(\frac{x''}{x}\right)'+\frac{B}{A}\left(\frac{x'}{x}\right)'\frac{y'}{y'}+\frac{D}{A}\left(\frac{x'}{x}\right)'=0\ \ldots\ldots\ldots(4)$

$$\left(\frac{x''}{x}\right)'+\frac{D}{A}\left(\frac{x'}{x}\right)'=-\frac{B}{A}\left(\frac{y'}{x}\right)'\frac{y'}{y}.$$

$$\frac{\left(\frac{x''}{x'}\right)'}{\frac{B}{A}\left(\frac{x'}{x}\right)'} + \frac{D}{B} = -\frac{y'}{y} \quad \ldots\ldots\ldots\ldots\ldots(5)$$

This equation is separated, so that both sides must be equal to a constant, λ. $y' + \lambda y = 0$

(i) $\frac{y'}{y} = -\lambda \quad \ldots\ldots\ldots\ldots\ldots(6)$

sub equation (6) in (5),

$$\left(\frac{x''}{x}\right)' + \frac{D}{B} = +x\frac{B}{A}\left(\frac{x'}{x}\right)'$$

$$\left(\frac{x''}{x}\right)' + \left(\frac{D}{B} - \lambda\right)\frac{B}{A}\left(\frac{x'}{x}\right)' = 0 \quad \ldots\ldots\ldots\ldots(7)$$

Integrating (7) $w \cdot r \cdot$ to x,

$$\left(\frac{x''}{x}\right) + \left(\frac{D}{B} - \lambda\right)\frac{B}{A}\left(\frac{x'}{x}\right) = -\beta \quad \ldots\ldots\ldots\ldots(8)$$

where β is a constant.

$$(3) \Rightarrow \frac{x''}{x} + \frac{B}{A}\frac{x'}{x}\left(\frac{-\lambda y}{y}\right) + \frac{C}{A}\left(\frac{-\lambda y'}{-y'}\right) + \frac{D}{A}\frac{x'}{x} +$$

$$\frac{E}{A}\left(\frac{-\lambda y}{x}\right) + \frac{F}{A} = 0.$$

$$\Rightarrow \frac{x''}{x} + \frac{B}{A}\frac{x'}{x}(a - \lambda) + \frac{C}{A}(\lambda^2) + \frac{D}{A}\left(\frac{x'}{x}\right) - \lambda\frac{E}{A} + \frac{F}{A} = 0$$

$$\Rightarrow \frac{x''}{x} + \frac{x'}{x}\left(-\frac{B}{A}\lambda + \frac{D}{A}\right) + \frac{C}{A}\left(\lambda^2 - \frac{\lambda E}{A} + \frac{F/A}{C/A}\right) = 0.$$

$$\Rightarrow \frac{x''}{x} + \frac{x'}{x'}\frac{B}{A}\left(-\lambda + \frac{D/A}{B/A}\right) + \frac{C}{A}\left(\lambda^2 - \lambda\frac{E}{C} + F/C\right) = 0. \quad \ldots\ldots\ldots\ldots(9)$$

$$\Rightarrow \frac{x''}{x} + \frac{x'}{x}\frac{B}{A}\left(-\lambda + \frac{D}{B}\right) = -\frac{C}{A}\left(\lambda^2 - \lambda\frac{E}{C} + \frac{F}{C}\right). \quad \ldots\ldots\ldots\ldots(10)$$

combine (8) and (10),

$$\beta = \left(\lambda^2 - \frac{E}{c}\lambda + \frac{F}{C}\right)\frac{C}{A}$$

3.2. The Vibrating String Problem:

consider $u_{tt} - c^2 u_{xx} = 0 \qquad 0 < x < l, t > 0 \quad \ldots\ldots\ldots..(1)$

$u(x,0) = f(x) \quad 0 \le x \le l \quad \ldots\ldots\ldots\ldots(2)$
$u_t(x,0) = g(x) \quad 0 \le x \le l \quad \ldots\ldots\ldots\ldots(3)$
$u(0,t) = 0 \quad t \ge 0 \quad \ldots\ldots\ldots\ldots(4)$
$u(l,t) = 0 \quad t \geqslant 0 \quad \ldots\ldots\ldots\ldots(5)$

where f and g are the initial displacement and initial velocity. respect.

By the method of separation of variables, we assume a solution in the form

$u(x,t) = x(x)T(t) \neq 0 \quad \ldots\ldots\ldots\ldots (6)$

(1) $\Rightarrow x\tau'' = -c^2 x'' T$

$\frac{x''}{x} = \frac{1}{c^2}\frac{T''}{T} \quad \ldots\ldots\ldots.(7) \cdot (\because \times\, T \neq 0).$

$u = xT$
$u_{tt} = x\tau''.$
$u_{xx} = x''T.$

Here L.H.S depend only X and R.H.S depend only t.

$\therefore \ \frac{x''}{x} = \frac{1}{c^2}\frac{T''}{T} = \lambda.$

where λ is a separation constant.

$\Rightarrow \frac{x''}{x} = \lambda$
$\Rightarrow x'' - \lambda x = 0 \quad \ldots\ldots\ldots\ldots (8)$
$\frac{1}{c^2}\frac{T''}{T} = \lambda$
$\tau'' - \lambda c^2 T = 0 \quad \ldots\ldots\ldots\ldots (9)$

sub equation (4) in (6)

$u(0,t) = x(0)T(t) = 0. \ (\because T(t) \neq 0)$
$\Rightarrow x(0) = 0 \rightarrow$ (10.

sub equation (5) in (6),

$u(l,t) = x(l)T(t) = 0$
$\Rightarrow x(l) = 0 \ (\because T(t) + 0) \rightarrow$ (11).

Now to find the value of $x(x)$ we have to solve

$x'' - \lambda x = 0$
$x(1) = 0$

we investigate the three cases.

$\lambda > 0,\ \lambda = 0,\ \lambda < 0.$

case (i) :- $\lambda > 0.$

The differential equation is $x'' - \lambda x = 0.$

The characteristic equations $m^2 - \lambda = 0$

$\Rightarrow m^2 = \lambda.$
$\Rightarrow m = \pm\sqrt{\lambda}.$

The general solution is of the form

$x(x) = Ae^{-\sqrt{\lambda}x} + Be^{\sqrt{\lambda}x}$.

where A and B are arbitrary constants.

Apply boundary and.

$$\left.\begin{aligned} x(0) = 0 &\Rightarrow A + B = 0 \\ x(l) = 0 &\Rightarrow Ae^{-\sqrt{\lambda}l} + Be^{\sqrt{\lambda}l} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (13).$$

Hence the general solution $x(x) = 0$.

$\therefore$ The solution is trivial.

case (ii) : $\lambda = 0$.

$\therefore \; x'' = 0$.

The characteristic equation is $m^2 = 0$.

$m = 0, m = 0$.

$\therefore$ The general solution $x(x) = A + B(x)$.

Applying boundary conditions:

$x(0) = 0 \Rightarrow A = 0$

$x(l) = 0 \Rightarrow A + Bl \; = 0 \ldots\ldots\ldots\ldots\ldots (14)$.

$Bl \; = 0 \; (\because A = 0)$.

$\Rightarrow B \; = 0 \; (\because l \neq 0)$

Hence $A = B = 0$.

$\therefore$ The solution is trivial.

Case (iii) $\lambda < 0$.

The general solution assumes the form

$x(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$

$x(0) = A + 0 = 0$.

$\therefore A = 0$

$x(l) = 0 \Rightarrow A\cos\sqrt{-\lambda}l + B\sin\sqrt{-\lambda}l = 0$

For $B\sin\sqrt{-\lambda}l = 0 \; (\because A = 0)$

For nontrivial sol, $B \neq 0$ ($B = 0$, then the $\sin\sqrt{-\lambda}\ell = 0 \quad \ldots\ldots\ldots(16)$

$\Rightarrow +\sqrt{-\lambda}l = n\pi$ for $n = 1,2,3, \ldots.$

$\Rightarrow -\lambda_n = \left(\frac{n\pi}{l}\right)^2$.

For this infinite set of discrete values of λ, the problem has a non-brivial solution. These values of λ_n are called the reign values of the problem.

Sub in equation (16), $\sin\left(\frac{n\pi}{\lambda}\right)x, n = 1,2,3, ...$

Hence the solution is $x_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$

For $\lambda = \lambda_n$, the general solution of (9) is.

$T_n(t) = C_n \cos\left(\frac{n\pi c}{l}\right)t + D_n \sin\left(\frac{n\pi l^c}{l}\right)t$ .... (17)

where C_n, D_n are arbitrary constants.

Hence $u_n(x,t) = x_n(x)T_n(t)$.

$$= \left(B_n \sin\left(\frac{n\pi x}{l}\right)\right)\left(c_n \cos\left(\frac{n\pi c}{l}\right)t + D_n \sin\left(\frac{n\pi c}{l}\right)t\right).$$

$$= \left(a_n \cos\left(\frac{n\pi c}{l}\right)t + b_n \sin\left(\frac{n\pi c}{l}\right)t\right)\sin\left(\frac{n\pi x}{l}\right).$$

where $a_n = B_n C_n$ and $b_n = B_n D_n$.

the infinite series,

$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi c}{l}\right)t + b_n \sin\left(\frac{n\pi c}{l}\right)t\right)$ .... (18)

is converges and twice continuously differentiable.

Now w.r.to x and t.

Now to find a_n and b_n.

Applying boundary conditions, $u(x,0) = f(x) \Rightarrow \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi nx}{l}\right) = f(x)$ ....(19)

Differentiation equation (18) w.r.to ' t ',

$$u_t = \sum_{n=1}^{\infty} \left(-a_n \sin\left(\frac{\pi nc}{l}\right)t\phi \times \frac{n\pi c}{l} + b_n \cos\left(\frac{n\pi c}{l}\right)t\right.$$

$$\times \left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$= \sum_{n=1}^{\infty} \frac{n\pi c}{l}\left(-a_n \sin\left(\frac{n\pi c}{l}\right)t + b_n \cos\left(\frac{n\pi c}{l}\right)t\right)$$

$$\sin\left(\frac{n\pi x}{l}\right)$$

$$u_t(x,0) = g(x) \Rightarrow \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi nx}{l}\right)\left(\frac{n\pi c}{l}\right) = g(x).$$

$f(x)$ and $g(x)$ represented as Fourier sine series.

∴ The coefficients are gi by,

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

∴ The solution of the vibrating string is

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi c}{l}t + b_n\sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi x}{l}$$

where $a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$ and.

$$b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

Example:1 (The plucked string)

consider a stretched string fixed at both sides suppose the string is raised to a high h at $x = a$, and then released.

The string oscillated freely The initial conditions are written as

$$u(x,0) = f(x) = \begin{cases} \dfrac{hx}{a}, 0 \le x \le a. \\ \dfrac{h(l-x)}{l-a}, a \le x \le l. \end{cases}$$

$$u_t(x,0) = g(x) = 0.$$

To find solution $u(x,t)$.

$$W\cdot k\cdot T, u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi c}{l}t + b_n\sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi x}{l}$$

where $a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$

$$b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

Since $g(x) = 0 \Rightarrow b_n = 0.$

Now

$$a_n = \frac{2}{l}\int_0^l \left(\frac{hx}{a} + \frac{h(l-x)}{l-a}\right)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{2}{l}\int_0^a \frac{hx}{a}\sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_a^l \frac{h(l-x)}{l-a}\sin\frac{n\pi x}{l}\,dx.$$

consider,

$$\frac{2}{l}\int_0^a \frac{hx}{a}\sin\frac{n\pi x}{l}\,dx = \frac{2h}{la}\int_0^a x\sin\frac{\text{nnx}}{l}\,dx.$$

[let $u = x$, $du = dx$

$dv = \sin n\pi x.$

$v = -\cos\frac{n\pi x}{l} \times \frac{l}{n\pi}\Big].$

$$= \frac{2h}{la}\left[\left(x\left[-\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right)_0^a + \int_0^a \cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\,dx\right]$$

$$= \frac{lh}{la}\left(-\frac{al}{n\pi}\cos\frac{n\pi a}{l} + \frac{l}{n\pi}\left(\sin\frac{n\pi x}{l}\right)_0^a \frac{l}{n\pi}\right)$$

$$= \frac{2h}{la}\left[-\frac{al}{n\pi}\cos\frac{n\pi a}{l} + \frac{l^2}{n\pi^2}\sin\frac{n\pi a}{l}\right].$$

$$= \frac{lh}{a}\left[\frac{l}{n^2\pi^2}\sin^2\frac{n\pi a}{l} - \frac{a}{n\pi}\cos\frac{n\pi a}{l}\right].$$

consider,

$$\frac{2}{l}\int_a^l \frac{h(l-x)}{l-a}\sin\frac{n\pi x}{l}\,dx = \frac{2h}{l(l-a)}\int_a^l (l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2h}{l(l-a)}\Bigg(\Bigg[(l-x)\left(-\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right)\Bigg]_a^l + \int_a^l \cos\frac{n\pi x}{l}\cdot\frac{l}{\pi n}(-dx)\Bigg)$$

$$= \frac{2h}{l(l-a)}\Bigg[\Bigg(\frac{l(l-a)}{n\pi}\left(\cos\frac{n\pi a}{l}\right)\Bigg] - \left(\sin\frac{n\pi x}{l}\right)_a^l$$

$$= \frac{2h}{l(l-a)}\Bigg(\frac{l(l-a)}{n\pi}\left(\frac{\cos n\pi a}{l}\right) - \sin\frac{l^2}{\pi^2 n^2}\Bigg].\frac{l^2}{\pi^2 n^2} + \sin\frac{n\pi a}{l}\cdot\frac{l^2}{\pi^2 n^2}\Bigg)$$

$$= \frac{2n}{l(l-a)}\left(\sin\frac{n\pi a}{l}\cdot\frac{l^2}{\pi^2 n^2} + \frac{l(l-a)}{n\pi}\cos\frac{n\pi a}{l}\right).$$

$$\therefore a_n = \frac{2hl}{an^2\pi^2}\sin\frac{n\pi a}{l} - \frac{2h}{n\pi}\cos\frac{n\pi a}{l} + \frac{2hl}{(l-a)\pi^2 n^2}\sin\frac{n\pi a}{l} + \frac{2h}{n\pi}\cos\frac{n\pi a}{l}$$

$$= \frac{2hl}{n^2\pi^2}\sin\frac{n\pi a}{l}\left(\frac{1}{a} + \frac{1}{l-a}\right)$$

$$= \frac{2hl}{n^2\pi^2}\sin\frac{n\pi a}{l}\left(\frac{l-a+a}{a(l-a)}\right).$$

$$\therefore a_n = \frac{2hl^2}{n^2\pi^2}\sin\frac{n\pi a}{l}\,\frac{l}{a(l-a)}.$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{2hl^2}{\pi^2 a(l-a)}\frac{1}{n^2}\frac{\sin n\pi a}{l}\frac{\cos u\pi c}{l} + t\Bigg)\sin\frac{n\pi x}{l}.$$

$$= \frac{2hl^2}{\pi^2 a(l-a)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\cos\frac{\pi n c}{l}t.$$

Example 2:(The stuck string)

Consider the string with no initial displacement. Let the string be struck at $x = a$. so the initial velocity is in by,

$$u_+(x,0) = \begin{cases} \dfrac{v_0}{a}x, & 0 \le x \le a. \\ v_0\dfrac{(l-x)}{l-a}, & a \le x \le l. \end{cases}$$

Since the string with no initial displacement, $u(x,0) = 0$.

(i) $f(x) = 0$.

$\Rightarrow a_n = 0$.

Now $b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}dx.$

$$= \frac{2}{n\pi c}\left[\int_0^a \frac{v_0}{a}x\frac{\sin^n \pi x}{l}dx + \int_a^l \frac{(l-x)}{l-a}v_0\frac{\sin n_x x}{l d d_2}\right]$$

consider,

$$\int_0^a \frac{v_0}{a}x\frac{\sin n\,\pi x}{l}dx = \frac{v_0}{a}\int_0^a x - \frac{\sin n\pi x}{l}dx.$$

$$= \frac{v_0}{a}\left(\left[-x\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right]_0^a + \int_0^a \cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}dx\right)$$

$$= \frac{v_0}{a}\left(-a\cos\frac{n\pi a}{l}\cdot\frac{l}{n\pi} + \frac{l}{n\pi}\cdot\frac{l}{n\pi}\left[\sin\frac{n\pi x}{l}\right]_0^a\right)$$

$$= \frac{V_0 l}{an\pi}\left[\frac{l}{n\pi}\sin\frac{n\pi a}{l} - a\cos\frac{n\pi a}{l}\right].$$

consider,

$$\int_a^l \frac{(l-x)}{l-a}v_0\sin\frac{n\pi x}{l}dx = \frac{v_0}{l-a}\int_a^l (d-x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{v_0}{l-a}\left(\left[-(l-x)\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right]_a^l + \int_a^l \frac{\cos n\pi x}{l}\cdot\frac{l}{n\pi}(-x)\right)$$

$$= \frac{v_0}{l-a}\left[(l-a)\cos\frac{n\pi a}{l}\cdot\frac{l}{n\pi} - \left[\sin\frac{n\pi x}{l}\cdot\frac{l^2}{n^2\pi^2}\right]_a^l\right]$$

$$= \frac{v_0}{l-a}\left[(l-a)\cos\frac{n\pi a}{l} - \frac{l}{n\pi} - \sin n\pi\frac{l^2}{n^2\pi^2} + \sin\frac{n\pi a}{l}\frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{V_0 l}{(l-a)n\pi}\left[(l-a)\cos\frac{n\pi a}{l} + \sin\frac{n\pi a}{l}\cdot\frac{l}{n\pi}\right].$$

$$\therefore b_n = \frac{2}{n\pi c}\left[\frac{v_0 l}{an\pi}\left(\frac{l}{n\pi}\sin\frac{n\pi a}{l} - a\cos\frac{n\pi a}{l}\right) + \right.$$

$$\left.\frac{V_0 l}{(l-a)n\pi}\left((l-a)\cos\frac{n\pi a}{l} + \sin\frac{n\pi a}{l}\cdot\frac{l}{n\pi}\right)\right]$$

$$= \frac{2v_0 l}{n^2\pi^2 c}\left[\frac{l}{an\pi}\sin\frac{n\pi a}{l} - \cos\frac{n\pi a}{l} + \frac{\cos n\pi a}{l} + \sin\frac{n\pi a}{l}\frac{l}{(l-a)n\pi}.\right]$$

$$= \frac{2v_0 l}{n^2\pi^2 c}\frac{l}{n\pi}\left(\sin\frac{n\pi a}{l}\right)\left(\frac{1}{a} + \frac{1}{l-a}\right)$$

$$= \frac{2v_0 l^2}{n^3\pi^3 c}\sin\left(\frac{n\pi a}{l}\right)\left(\frac{l-a+a}{(l-a)a}\right).$$

$$= \frac{2v_0 l^3}{\pi^3 ca(l-a)}\frac{1}{n^3}\sin\left(\frac{n\pi a}{l}\right).$$

Hence the displacement of the struck string is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2v_0 l^3}{n^3 ca(l-a)}\frac{1}{n^3}\sin\left(\frac{n\pi a}{l}\right)\sin\left(\frac{\pi nc}{l}t\right)$$

$$\sin\frac{n\pi x}{l}.$$

$$= \frac{2v_0 l^3}{\pi^3 ca(l-a)}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\sin\frac{n\pi c}{l}t.$$

3.3. Existence and uniqueness of solution of the vibrating string problem:

We know that, $u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi c}{l}t + b_n\sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi x}{l}$.

To show that $u(x,t)$ is a solution of the vibrating String problem.

Let $u_1(x,t) = \sum_{n=1}^{\infty} a_n\cos\frac{n\pi c}{l}t\sin\frac{n\pi x}{l}$ with $g(x)$

$$u_2(x,t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi c}{l} + \frac{\sin n\pi x}{l}, f(x)_2$$

Now to show u_1 & u_2 are solution of the problem

To Prove: u_1 is a solution of $u_{tt} = c^2 u_{xx}$..

i) $u(x,0) = f(x)$.

ii) $u_t(x,0) = g(x)$.

ii $u(0,t) = 0$.

iv) $u(l,t) = 0$.

we assume that $f(x)$ and $f'(x)$ are continuous on $[0,l]$ and $f(0) = f(l) = 0$.

$\Rightarrow$ The series of the fo $\sum_{n=1}^{n} a_n \sin\frac{n\pi x}{l}$ convergences, absolutely & uniformly on the interval $[0,l]$

$$u_1(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi c}{l} t \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} a_n \left[\frac{1}{2}\sin\frac{n\pi}{l}(x-ct) + \frac{1}{2}\sin\frac{n\pi}{l}(x+ct)\right]$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}(x-ct) + \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}(x+ct) \quad \ldots\ldots\ldots\ldots(1)$$

Define $F(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$ and $F(x)$ is the periodic extension of $f(x)$.

(i.e.) $F(x) = f(x)\ 0 \le x \le l$.

$F(-x) = -F(x) \ \Leftrightarrow x \forall x$.

$F(x \pm 2l) = F(x)$

(1) $\Rightarrow u_1(x,\text{t}) = \frac{1}{2}[F(x-ct) + F(x+ct)] \ \ldots\ldots\ldots(2)$

To prove (i): $u_1(x,0) = \text{f(x)}$

Applying $x = x$ and $t = 0$ in (2).

$$u_1(x,0) = \frac{1}{2}[F(x) + F(x)].$$

$$= \frac{2}{2}F(x) = F(x)$$

$$= f(x)\ 0 \le x \le l$$

(F is periodic extension of $f(x)$)

To prove: (ii) $\frac{\partial u_1}{\partial t}(x,0) = g(x) = 0$

differentiating (2) $w \cdot r.$ to ' t '.

$\Rightarrow \frac{\partial u_1(x,t)}{\partial t} = \frac{1}{2}[F'(x-ct)(-c) + F'(x+ct)\cdot c] \quad \ldots\ldots\ldots\ldots (3)$

Now $\frac{\partial u_1(x,0)}{\partial t} = \frac{1}{2}[F'(x)(-c) + cF'(x)]$

$$= \frac{F'(x)}{2}(-c+c) = 0.$$

T.P:- (iii) $u_1(0,t) = 0$.

put $x = 0, t = t$ in equation (2).

$u_1(0,t) = \frac{1}{2}[F(-ct) + F(ct)].$

$= \frac{1}{2}[-F(ct) + F(ct)] = 0.$

$To\ prove\ (iv): u_1(l,t) = 0.$

put $x = l, t = t$ in equation (2).

$u_1(l,t) = \frac{1}{2}[F(l-ct) + F(l+ct)].$

$= \frac{1}{2}[F(-l-ct) + F(l+ct)]\ [\because F(x \pm 2l) = F(x)$

$= \frac{1}{2}[-F(l+ct) + F(l+ct)]\ \Rightarrow F(l-ct) = F(l-ct)$

$= 0.\ \ = F(-l-ct).$

∴ The boundary conditions and initial ends are satisfied.

Now To prove: $\frac{\partial^2 u_1}{\partial t^2} = c^2 \frac{\partial^2 u_1}{\partial x^2}$.

Let f'' be continuous on $[0,l]$ and let $f''(0) = f''(l) = 0$. Then F'' exists and is continuous everywhere. therefore, we differentiation equation (3) w.r.to ' t'.

$\Rightarrow \frac{\partial^2 u_1}{\partial t^2} = \frac{1}{2}[F''(x-ct)c^2 + F''(x+ct)c^2].$

$= \frac{c^2}{2}[F''(x-ct) + F''(x+ct)] \ldots \ldots \ldots \ldots \ldots (4)$

Now differentiation w.r.to x,

$\frac{\partial u_1}{\partial x} = \frac{1}{2}[F'(x-ct) + F'(x+ct)].$

again diff $w \cdot r \cdot$ to x,

$\frac{\partial^2 u_1}{\partial x^2} = \frac{1}{2}[F''(x-ct) + F''(x+ct)] \ldots \ldots \ldots \ldots (5).$

sub in eqn (5) in (4).

$\frac{\partial^2 u_1}{\partial t^2} = c^2 \frac{\partial^2 u_1}{\partial x^2}$

∴ u_1 satisfies the wave en.

2.) To prove: u_2 is the solution of the wave ign.

where $\varphi_2(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi c}{l} t \sin\frac{n\pi x}{l}$ ……….(6)

with $f(x) = 0$

Let g and g' be continuous on $[0,l]$ and let $g(0) = g(l) = 0$. Then the function

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n\left(\frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l}.$$

converges absolutely and uniformly in [0, l]

Let $c_n = \left(\frac{n\pi c}{l}\right) b_n$.

$\Rightarrow b_n = \left(\frac{l}{n\pi c}\right) c_n$.

Sub equation (6) in, $u_2(x,t) = \sum_{n=1}^{\infty}\left(\frac{l}{n\pi c}\right) c_n \sin\frac{n\pi c}{l} t \frac{\sin n\pi x}{l}$.

$$= \frac{l}{\pi c}\sum_{n=1}^{\infty}\frac{c_n}{n}\frac{\sin n\pi c}{l} t \frac{\sin n\pi x}{l}.$$

Diff $w \cdot r$. to ' t ',

$$\frac{\partial u_2}{\partial t} = \frac{\ell}{\pi c}\left(\sum_{n=1}^{\infty}\frac{c_n}{n}\frac{\cos n\pi c}{l} + \sin\frac{n\pi x}{l}\right) \times \frac{\pi\pi k}{k}$$

$$= \sum_{n=1}^{\infty} c_n \frac{\cos n\pi c}{l} + \sin\frac{n\pi x}{l} \quad \ldots\ldots\ldots\ldots\ldots(7)$$

$$\frac{\partial u^2}{\partial t} = \frac{1}{2}\sum_{n=1}^{\infty}\left(c_n \sin\left(-\frac{n\pi c}{l}t + \frac{\sin n\pi x}{l}\right) + c_n \sin\left(\frac{n\pi c}{l}t + \sin\frac{n\pi x}{l}\right)\right)$$

$= \frac{1}{2}\sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{l}(x - ct) + \frac{1}{2}\sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{l}(x + ct)$ ....(8)

The series are absolutely and uniformly convergent because of the assumption G.

Hence (7) & (8) are converge absolutely and uniformly on $[0, c]$.

Let $G(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}$.

be the odd periodic extension of the function $f(x)$.

i.) $G(x) = g(x)\ 0 \le x \le l$
$G(-x) = -G(x)\ \forall x.$

$G(x + 2t) = G(x)$

Equation (8) $\Rightarrow \frac{\partial u_2}{\partial t} = \frac{1}{2}[G(x - ct) + G(x + ct)]$ ....(9)

Integrating, $w.r.$ to.$'t'$ '

$$U_2 = \frac{1}{2}\int_0^t \big(G(x-ct') + G(x+ct')\big)dt' \rightarrow (*).$$

$$= \frac{1}{2}\int_0^t G(x-ct')dt' + \frac{1}{2}\int_0^t G(x+ct')dt'.$$

$$= \frac{1}{2}\left[\frac{G(x-ct')}{-c}\right]_0^t + \frac{1}{2}\left[\frac{G(x+ct')}{c}\right]_0^t.$$

$$= \frac{1}{2c}[-G(x-ct) + G(x) + G(x+ct) - G(x)].$$

$$= \frac{1}{2c}[G(x+ct) - G(x-ct)].$$

$= \frac{1}{2c}\int_{x-ct}^{x+ct} G(\tau)d\tau$ ………….(10)

i) To prove: $u_2(x,0) = 0$:

sub $x = x, t = 0$ in

$$u_2(x,0) = \frac{1}{2c}\int_x^x G(\tau)d\tau$$

$$= 0.$$

(ii)

To prove: $u_t(x,0) = g(x)$.

$$\frac{\partial u_2}{\partial t}(x,0) = \frac{1}{2}[G(x) + G(x)].$$

From (9), $= \frac{2G(x)}{2} = G(x)$

$= g(x)$

(iii) To prove: $u_2(0,t) = 0$

From $(*)$, $U_2(0,t) = \frac{1}{2}\int_0^t G(-ct') + G(ct')dt'$.

$$= \frac{1}{2}\int_0^t G(-ct')dt' + 1/2\int_0^t G(ct')dt$$

$$= -\frac{1}{2}\int_0^t G(ct')dt' + 1/2\int_0^t G(ct')dt.$$

$= 0.$

(iv) To prove: $U_2(c,t) = 0$

$$(t) \Rightarrow u_2(l,t) = \frac{1}{2}\int_0^t G(l-ct')dt' + 1/2\int_0^t G(l+ct')dt'.$$
$$= 1/2\int_0^t G(-l-ct')dt' + 1/2\int_0^t G(l+ct')dt'$$
$$= -1/2\int_0^t G(l+ct')dt' + 1/2\int_0^t G(l+ct')dt'.$$
$$= 0.$$

Finally to prove:- $\frac{\partial^2 u_2}{\partial t^2} = c^2\frac{\partial^2 u_2}{\partial x^2}$.

Since g' is continuous on $[0,l], G'$ exists so that, by equation (9),

$$\frac{\partial^2 u_t}{\partial t^2} = \frac{1}{2}[G'(x-ct)(-c) + G'(x+ct)(+c)].$$

Now differentiating u_2 w.r.to ' x '

$$\frac{\partial u_2(x,t)}{\partial x} = \frac{l}{\pi c}\sum_{n=1}^{\infty}\frac{c_n}{n}\sin\frac{n\pi c}{l} + \cos\frac{n\pi x}{l}\cdot\frac{n\pi}{l}.$$
$$= \frac{1}{c}\sum_{n=1}^{\infty} c_n \sin\frac{n\pi ct}{l}\cos\frac{n\pi x}{l}.$$
$$= \frac{1}{c}\sum_{n=1}^{\infty} c_n\left[1/2\left(\sin\left(\frac{n\pi ct}{l}+\frac{n\pi x}{l}\right) + \sin\left(\frac{n\pi ct}{l}-\frac{n\pi x}{l}\right)\right]\right.$$
$$= \frac{1}{2c}\sum_{n=1}^{\infty} c_n\left[\sin\left(\frac{ct+x}{l}\right)\frac{n\pi}{l} + \sin(ct-x)\frac{n\pi}{l}\right].$$
$$= \frac{1}{2c}\sum_{n=1}^{\infty} c_n\left[-\sin(x-ct)\frac{n\pi/l}{l} + \sin(x+ct)\frac{n\pi}{l}\right].$$
$$= \frac{1}{2c}[-G(x-ct) + G(x+ct)] \text{ by def of } G(x).$$

Differentiating, $w\cdot r\cdot$ to ‘x '

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{1}{2c}[-G'(x-ct) + G'(x+ct)]$$

$$2c\frac{\partial^2 u_2}{\partial x^2} = -G'(x-c) + G_1'(x+ct) \quad \text{............ (12)}$$

sub eqn (12) in (11)

$$\frac{\partial^2 u_t}{\partial t^2} = \frac{c}{2}2c\frac{\partial^2 u_2}{\partial x^2}$$
$$\Rightarrow \frac{\partial^2 u_t}{\partial t^2} = c^2\frac{\partial^2 u_2}{\partial x^2}.$$

Theorem 1: (uniqueness theorem)

There exists almost one solution of the wave equation $u_{tt} = c^2 u_{xx}, 0 < x < 1, t > 0.$

satisfying the initial condition, $u(x,0) \quad = f(x),\ 0 \leq x \leq l.$
$u_t(x,0) \quad = g(x),\ 0 \leq x \leq l.$

and the boundary ends, $u(0,t) = 0,\ t \geq 0.$
$u(l,t) = 0,\ t \geq 0.$

where $u(x,t)$ is a twice continuously differentiable fo w.r. to both x and t.

Proof:

suppose there are two solutions u_1 and u_2.

Let $v = u_1 - u_2$.

$\Rightarrow v(x,t)$ is the sol of the problem

$$v_{tt} = c^2 v_{xx}, 0 < x < l, t > 0.$$
$$v(0,t) = u_1(0,t) - u_2(0,t)$$
$$= 0 - 0 = 0\,, \quad t \geq 0$$
$$v(l,t) = u_1(l,t) - u_2(l,t)$$
$$= 0 - 0 = 0\ , \quad t \geq 0$$

$$v(x,0) = u_1(x,0) - \mu_2(x,0)$$
$$= f(x) - f(x) = 0\ 0 \leqslant x \leqslant l$$
$$v_t(x,0) = \frac{\partial u_t}{\partial t}(x,0) - \frac{\partial u_2}{\partial t}(x,0).$$
$$= g(x) - g(x) = 0.\ 0 \leqslant x \leqslant l.$$

To prove: $V(x,t) = 0$

Consider the $fn, I(t) = \frac{1}{2}\int_0^l (c^2 V_x^2 + V_t{}^2)dx$

represent the total energy of vibrating string at time. the total energy of vibrating string at time y since the function $V(x,t)$ is twice continuously differentiable.

We differentiate I(t) w.r.to ‘t’

$$\frac{dx}{dt} = \frac{1}{2}\int_0^l (x^2 v_x v_{xt} + 2v_t v_{tt})dx.$$
$$= \int_0^l (c^2 v_x v_{xt} + v_t v_{tt})dx.$$

Consider $\int_0^l c^2 v_x v_{xt} dx = [c^2 v_x v_t]_0^l - \int_0^l c^2 v_t v_{2x} dx$

$$= -\int_0^l c^2 v_t v_{xx} dx$$

$$\therefore \frac{dI}{dt} = -\int_0^l c^2 v_t v_{xx} dx + \int_0^l v_t v_{tt} dx$$

$$= \int_0^l v_t \left(v_{tf} - \grave{c}^2 v_{xx}\right) dx$$

$= 0$ ($\because v_{tt} - c^2 v_{xx}) = 0$

$\therefore I(t) =$ constant $= C$

Since $v(x,0) = 0 \Rightarrow v_x(x,0) = 0$ and $v_t(x,0) = 0$.

$$I(0) = \frac{1}{2}\int_0^l (e^2 {v_{xt}}^2 + {v_t}^2) dx = c.$$
$$= 0.$$

$\therefore I(t) = 0$, whenever $v_x = 0$ & $v_t = 0$ for $t > 0$.

$\therefore V(x,t) =$ constant .

since $v(x,0) = 0 \Rightarrow v(x,t) = 0$.

$\therefore u_1(x,t) - u_2(x,t) = 0.$
$\Rightarrow u_1(x,t) = u_2(x,t).$

$\therefore$ The solution is unique.

3.4.

3.4. The Heat conduction problem:

We consider a homogeneous rod of length l. The rod is thin so the heat is distributed equally over the cross section at time t. The surface of the rod is insulated, there is no heat loss through the boundary.

$$u_t = k u_{xx},\ 0 < x < l,\ t > 0 \ldots\ldots\ldots.(1).$$
$u(0,t) = 0, \quad t \geq 0.$
$u(x,t) = 0, \quad t \geq 0.$

Assume the solution,

$u(x,t) = x(x), T(t) \neq 0.$ $\quad \left[\begin{array}{l} u_t = xT' \\ u_{xx} = x''T \end{array} \right]$

$$(1) \Rightarrow \begin{array}{l} xT' = kx''T \\ \dfrac{x''}{x} = \dfrac{T'}{KT} \end{array}$$

differentiation w.r.to ' x ',

$$\frac{\partial}{\partial x}\left(\frac{x''}{x}\right) = 0$$

Integrating, $\frac{x''}{x} = -\alpha^2, \ldots\ldots\ldots\ldots..(2) \alpha$ is $a + ve$ constant.

$\frac{T'}{KT} = -\alpha^2 \Rightarrow T^1 + \alpha^2 KT = 0 \quad \ldots\ldots\ldots (3)$

From the boundary conditions,

$u(0,t) = 0 \Rightarrow u(0,t) = x(0) + (t) = 0.$

$\Rightarrow x(0) = 0, T(t) \neq 0.$

$u(l,t) = 0 \Rightarrow u(l,t) = x(l)T(t) = 0.$

$\Rightarrow x(l) = 0; T(t) \neq 0.$

Equation (2) $\Rightarrow\ x'' + \alpha^2 x = 0.$

The characteristic equation $D^2 + \alpha^2 = 0.$

$D^2 = -\alpha^2$

$D = \pm\alpha i^\circ$

The solution of $x(x)$ is

$x(x) = A\cos\alpha x + B\sin\alpha x.$

$x(0) = 0 \Rightarrow x(0) = A = 0$

$\Rightarrow A = 0$

$x(1) = 0 \Rightarrow x(1) = A\cos\alpha l + B\sin\alpha l = 0.$

$= 0 + B\sin\alpha l = 0.$

$\Rightarrow\ B\sin dl = 0.$

If $B = 0$, the solution is trivial,

For nontrivial solution, $\sin\alpha l = 0.$

$\alpha l = n\pi.$

$\alpha = \frac{n\pi}{l}., n = 1,2, \ldots.$

$\therefore$ The solution $x_n(x) = B_n \sin\left(\frac{n\pi}{l}\right)x$; Here $A = 0$

Now (3) $\Rightarrow T^1 + \alpha^2 kT = 0$

The characteristic equation is $D + \alpha^2 k = 0.$

$D = -\alpha^2 k.$ (only one root)

$T(t) = ce^{-\alpha^2 kt}.$

sub $\alpha = \frac{n\pi}{l},$

$\Rightarrow T_n(t) = c_n e^{-\left(\frac{n\pi}{l}\right)^2 kt}.$

$\Rightarrow u_n(x,t) = x_n(x)Tn(t)$

$= B_n \sin\left(\frac{n\pi}{l}\right)x c_n e^{-\left(\frac{n\pi}{l}\right)^2 kt}.$

$= a_n\ e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l}$

where $a_n = B_n C_n$ is an arbitrary constant. we formally form a series.

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l} \ldots\ldots..(4)$$

Now, from initial condition $\Rightarrow u(x,0) = f(x)$

From equation (4) $\Rightarrow n(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{ntx}{l} = f(x)$

This is true if $f(x)$ can be represented by a Fourier sine series with Fourier coefficients.

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx.$$

Hence $\quad u(x,t) = \sum_{n=1}^{\infty} \left[\frac{Q}{l}\int_0^l f(\tau)\sin\frac{n\pi\tau}{l}d\tau\right] e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l}$

is the formal solution of the heat conduction problem.

Example 1:

suppose the initial temperature distribution is $f(x) = x(l-x)$. Find the solution

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{l}\int_0^l f(\tau)\sin\frac{n\pi\tau}{l}d\tau\right] e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l}.$$

$$= \sum_{n=1}^{\infty} \left[\frac{2}{l}\int_0^l (\tau)(l-\tau)\sin\frac{n\pi\tau}{l}d\tau\right] e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l}.$$

consider,

$$\int_0^l \tau(l-\tau)\sin\frac{n\pi\tau}{l}d\tau = \int_0^l (l\tau-\tau^2)\sin\frac{n\pi\tau}{l}d\tau.$$

Let $u = l\tau - \tau^2, dv = \sin\frac{n\pi\tau}{l}d\tau.$

$$du = (l-2\tau)d\tau,\ v = -\left(\frac{\cos n\pi\tau}{l}\right) \times \frac{l}{n\pi}$$

d.

$$\therefore \int_0^l \tau(l-\tau)\sin\frac{n\pi\tau}{l}d\tau = \left[-\frac{l}{n\pi}(l\tau-\tau^2)\cos n\frac{n\tau}{l}\right]_0^l + \frac{l}{n\pi}\int_0^l \frac{\cos n\pi\tau}{l}(l-2\tau)d\tau$$

$$= \frac{l}{n\pi}\int_0^l \frac{\cos n\pi\tau}{l}(l-2\tau)d\tau.$$

$$u = l - 2\tau\ dv = \cos\frac{n\pi\tau}{l}$$

$$du = -2d\tau \quad v = \frac{l}{n\pi}\cdot\sin\frac{n\pi\tau}{l}$$

$$= \frac{l}{n\pi}\left[(-l-2\tau)\frac{l}{n\pi}\sin\frac{n\pi\tau}{l}\right]_0^l + \int_0^l \frac{l}{n\pi}\sin\frac{n\pi\tau}{l}2d\tau$$

$$= \frac{2l^3}{n\pi^3}\left[-\cos\frac{n\pi\tau}{l}\right]_0^l$$

$$= \frac{2l^3}{n^3\pi^3}[-\cos n\pi + 1]$$

$$= \frac{4l^3}{n^3\pi^3}, \text{ where } n = 1,3,5, \ldots.$$

$$\therefore u(\mathrm{x}, t) = \sum_{n=1}^{\infty} \frac{2}{l} \times \frac{4l^2}{n^2\pi^3} e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l}$$

$$= \frac{8l^2}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^3} e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l}.$$

b) Suppose the temperature at one end of the rod is held constant (e) $u(l,t) = u_0, t \geq 0$.

$\therefore$ The problem becomes $u_t = ku_{xx}, 0 < x < t, t > 0$ $\quad u(0,t) = 0$, $u(l,t) = u_0$. $u(x,0) = f(x), 0 < x < l$.

Let $u(x,t) = v(x,t) + \frac{u_0 x}{l}$.

$\therefore v_t = ku_{xx}, \quad 0 < x < l \ \ , t > 0$

$v(0,t) = 0.$

$$v(l,t) = u(l,t) - u_0\frac{l}{l}.$$

$= u_0 - u_0$

$= 0.$

$$v(x,0) = u(x,0) - \frac{u_0 x}{l}$$

$$= f(x) - \frac{u_0 x}{l}, 0 < x < l$$

$$w \cdot k \cdot T, u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l f(\tau)\sin\frac{n\pi\tau}{l}d\tau\right] e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l \left(f - \frac{u_0\tau}{l}\right)\sin\frac{n\pi\tau}{l}d\tau\right] e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l} + \frac{u_0 x}{l}$$

3.5. Existence and uniqueness of solution of the heat conduction problem.

$$W \cdot k \cdot T, u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l} \ldots\ldots\ldots (1)$$

where $a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$ is the solution of the heat conduction problem.

To prove: The solution is satisfied if $f(x)$ is continuous on $[0,l]$ and $f(0) = f(l) = 0$ and $f'(\alpha)$ is piecewise continuous on $(0,l)$

since $f(x)$ is bounded.

$$|a_n| = \frac{2}{l}\left|\int_0^l f(x)\sin\frac{n\pi x}{l}dx\right|.$$

$$\leq \frac{2}{l}\int_0^l |f(x)|dx \leq c \text{ where } c \text{ is positive constant.}$$

Hence for any $t_0 > 0$.

$$\left|a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt}\sin\frac{n\pi x}{l}\right| \leq ce^{-\left(\frac{n\pi}{l}\right)^2 kt_0} \text{ when } t \geq t_0.$$

By ratio test, the series of the constant term $\exp\left[-\left(\frac{n\pi}{l}\right)^2 kt_0\right]$ converges.

Here by Weistrass M-test, the series.

$\sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt}\sin\frac{n\pi x}{l}$ is converges uniformly w.r.to x & t whenever $t \geq t_0$ f $0 \leq x \leq l$.

Differentiation equation (1) w.r.to ' t ',

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} - \left(\frac{n\pi}{l}\right)^2 k\sin\frac{\pi n x}{l}.$$

$$= -\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{l}\right)^2 ke^{-\left(\frac{n\pi}{l}\right)^2 xt}\sin\frac{\pi n x}{l} \quad \ldots\ldots\ldots\ldots (2)$$

Similarly,

$$\left|-a_n\left(\frac{n\pi}{l}\right)^2 ke^{-\left(\frac{n\pi}{l}\right)^2 kt}\sin\frac{n\pi x}{l}\right| \leqslant c\left(\frac{n\pi}{l}\right)^2 ke^{-\left(\frac{n\pi}{l}\right)^2 kt_0} \text{ when } t \geq t_0$$

$\therefore$ The series of the constant term $c\left(\frac{n\pi}{l}\right)^2 k\exp\left[-\left(\frac{n\pi}{l}\right)^2 kt_0\right.$

converges by ratio test.

Hence (2) converges uniformly.

$\therefore$ Similarly, we can differentiation equation (1) w.r.to $'x'$

$$u_x = \frac{\pi}{l}\sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt}\cos\frac{n\pi x}{l}.$$

again differentiation,

$$u_{xx} = -\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{l}\right)^2 e^{-\left(\frac{n\pi}{l}\right)^2 kt}\sin\frac{n\pi x}{l} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

From equation (2) and (3),

$u_t = ku_{xx}$.

Hence (1) is a solution of the one-dimensional heat equation in the region $0 \le x \le l; t > 0$

Now to prove, i) $u(0,t) = 0, t \geqslant 0$.

ii) $u(l,t) = 0, t \ge 0$.

iii) $u(x,0) = f(x)$,$0 \le x \le l$

Equation (1) representing the function $u(x,t)$ converges uniform, in the region

$0 \le x \le l, t > 0$.

A function represented by a uniformly convergent series of continuous function is continuous.

$\therefore u(x,t)$ is continuous at $x = 0$ & $x = l$.

(1) $\Rightarrow u(0,t) = 0.$
$u(l,t) = 0.$

$\therefore$ (i) & (ii) are satisfied.

iii) To prove: $u(x,0) = f(x)$.

Now assume that $f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$ is uniformly and absolutely convergent.

By Abel's test, the series formed by the product of the terms of uniformly convergent series $\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$ and the member of a uniformly bounded monotone sequence $e^{-\left(\frac{n\pi}{x}\right)^2 kt}$ convergent uniformly w.r.t

Hence $u(x,t)$ converges uniformly.

$$\therefore u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}.$$
$$= f(x),\ 0 \leqslant x \leqslant l.$$

Theorem 1: (uniqueness theorem)

Let $u(\mathrm{x},t)$ be a continuously differentiable function. If $u(x,t)$ satisfies the differential equation

$u_t = ku_{xx},\ 0 < x < l, t > 0.$

$u(x,0) = f(x),\quad 0 \le x \le l.$
$u(0,t) = 0,\qquad t \ge 0.$

$u(l,t) = 0, t \ge 0$ then it is unique.

Proof:

suppose there are two solution $u_1(x,t), u_2(x,t)$

Let $v(x,t) = u_1(x,t) - u_2(x,t)$

Then

$v_t = ku_{xx}, \quad 0 < x < l,\ t > 0 \quad \dots\dots\dots\dots (1).$
$v(0,t) = 0 \qquad t \geq 0$
$v(l,t) = 0 \qquad t \geq 0.$
$v(x,0) = 0 \qquad 0 \leq x \leq l.$

consider the function,

$$g(t) = \frac{1}{2k}\int_0^l v^2 dx.$$

Differentiation w.r.to ' t ',

$$J'(t) = \frac{1}{2k}\int_0^l 2vv_t dx.$$

$$= \frac{1}{k}\int_0^d vv_t dx = \frac{1}{k^x}\int_0^e vv_{xx} dx \text{ Let } u = v, dv = v_{xx}dx.$$

$(by\ (1)\ )$

$$= \int_0^l vv_{xx}dx. \qquad du = v_x,\ v = v_x.$$

$$= [vv_x]_0^l - \int_0^l v_x v_x dx$$

$$= v(l,t)v_x(l,t) - v(0,t)v_x(0,t) - \int_0^l v_x^2 dx.$$

$$= -\int_0^l V_x^2 dx \leq 0\ (\because V(l,t) = 0\ \&\ \ V(\text{l},t) = 0).$$

since $v(x,0) = 0; J(0) = \frac{1}{2k}\int_0^l v^2 dx = 0\ \& J'(t) \leqslant 0$

$\Rightarrow J(t)$ is decreasing for of t.

Thus $J(t) \leqslant 0$.

But by the definition $J(t)$

$\therefore J(t) = 0\ \forall t \geq 0$.

since $v(x,t)$ is continuous, $J(t) = 0$

$\Rightarrow v(x,t) = 0$ in $0 \leqslant x \leqslant l, t \geq 0$
$\therefore u_1(x,t) = u_2(x,t)$.

$\therefore$ The solution is unique:

3.6. The Laplace and Bean Equation:

Example 1:

Solve, $\nabla^2 u = 0,\ 0 < x < 0,\ \ 0 < y < b.$

$u(x,0) = f(x),\ 0 \le x \le 0.$
$u(x,b) = 0.$
$u_x(0,y) = 0.$
$u_x(a,y) = 0.$

Solution:

Let $u(x,y) = x(x)y(y)$ ... ... (1).

sub the Laplace equation $u_{xx} + u_{yy} = 0$

$$\therefore x''(x)y(y) + x(x)y''(y) = 0$$

$x''(x)y(y) = -x(x)y''(y).$

$$\frac{x''(x)}{x(x)} = -\frac{y''(y)}{y(y)}$$

differentiation $w \cdot r.$ to ' x ',

$$\frac{\partial}{\partial x}\left(\frac{x''(x)}{x(x)}\right) = 0$$

Integrating $w \cdot r \cdot t_0$ ' x ',

$$\frac{x''(x)}{x(x)} = \lambda$$

$\Rightarrow x''(x) - \lambda x(x) = 0$ ... ... (2)

$$-\frac{y''}{y} = \lambda$$

$\Rightarrow -y'' - \lambda y = 0$

$\Rightarrow y'' + \lambda y = 0$ ... ... (3)

Let $\lambda = -\alpha^2$,

(2) $\Rightarrow x'' + \alpha^2 x = 0$

The characteristic equation is $m^2 + \alpha^2 = 0.$

$m^2 = -\alpha^2.$
$m = \pm i\alpha.$

$\therefore x(x) = A\cos\alpha x + B\sin\alpha x.$

Applying boundary and,

$u_x(x,y) = x'(x)y(y)$
$u_x(0,y) = 0 \Rightarrow u_x(0;y) = x(0)y(y) = 0$
$u_x(a,y) = 0 \Rightarrow x'(a)y(y) = 0$
$\therefore x'(0) = 0$
$\Rightarrow x'(a) = 0.$

Now

$x'(0) = 0 \Rightarrow \ x'(x) = A\alpha \sin \alpha x + B\alpha \cos \alpha x$
$x'(0) = B\alpha \cos(0) = 0.$
$B\alpha = 0$
$B = 0 \sin \alpha > 0.$

Now $x'(a) = 0 \Rightarrow x'(a) = -A\ a \sin \alpha a = 0.$

if $A = 0$ we get a trivial solution,

$\therefore \sin \alpha a = 0.$
$\alpha a = n\pi$

Hence $x_n(x), \alpha = \dfrac{n\pi}{a}, n = 1,2, \ldots$

Hence $x_n(x) = A_n \cos\frac{n\pi x}{a} \rightarrow (4).$

Now to solve equation (3),

The characteristic equation is $m^2 - \alpha^2 = 0$

$m^2 = \alpha^2 \quad \Rightarrow \mathrm{m} = \pm\alpha$
$\therefore y(y) = ae^{\alpha y} + be^{-\alpha y}.$
$= a(\cosh \alpha y + \sinh \alpha y) + b(\cosh \alpha y - \sinh \alpha y).$
$= (a+b)\cosh \alpha y + (a-b)\sinh \alpha y.$
$= c\cosh \alpha y + D\sinh \alpha y.$

$$Y(y) = \sqrt{D^2 - C^2}\left[\frac{C}{\sqrt{D^2 - C^2}}\cosh \alpha y + \frac{D}{\sqrt{D^2 - C^2}}\sinh_\alpha y\right]$$

$$= \sqrt{D^2 - C^2}\left[\frac{C/D}{\sqrt{1 - C^2/D^2}}\cosh \alpha y + \frac{1}{\sqrt{1 - C^2/D^2}}\sinh_\alpha y\right].$$

$$= \sqrt{D^2 - C^2}\left[\cosh \alpha \sinh \sinh^{-1}\left(\frac{C/D}{\sqrt{1 - C/D^2}}\right) + \right.$$

$$\left.\sinh \alpha\, y \cosh \cosh^{-1}\left(\frac{1}{\sqrt{1 - \frac{C^2}{D^2}}}\right)\right]$$

$= \sqrt{D^2 - c^2}[\cosh \alpha y \sinh(\tan^{-1}(c/D)) +$
$\sinh xy \cosh(\tan^{-1}(C/D))].$

Lot $E = \sqrt{D^2 - C^2}$ and $G = \tan^{-1}(C/D)$.

$$= E(\sinh \alpha y \cosh G + \cosh \alpha y \sinh G)$$
$$= E\sin(\alpha y + G).$$
$$= E\sin \alpha(y + G/\alpha).$$

$$y(y) = E \sin \alpha(y + F) \text{ where } F = \frac{G}{\alpha} \quad \ldots\ldots\ldots\ldots.(5)$$

Apply boundary conditions in θ,

$$u(x, b) = 0 \Rightarrow x(x)y(b) = 0.$$
$$y(b) = 0 \Rightarrow y(b) = y(b) = 0 \ (\because x(x) \neq 0).$$
$$\Rightarrow \sin \alpha(b + F) = 0.$$
$$b + F = 0.$$
$$b = -F., E \neq 0.$$

for nontrivial solution, hence we have

$$u(x, y) = \frac{(b - y)}{b}\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a} \sin\frac{n\pi}{a}(y - b).$$

$$u(x, 0) = f(x) \Rightarrow \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a} \sin h\left(-\frac{n\pi b}{a}\right) = f(x).$$

Fourier series,

$$a_0 = \frac{2}{a}\int_0^a f(x)dx.$$

$$a_n = \frac{-2}{a\sin\frac{n\pi b}{a}}\int_a^a f(x)\cos\frac{n\pi x}{a}dx, n = 1,2,3, \ldots.$$

The formal solution is,

where $a^* = \frac{2}{a}\int_0^a f(x)\cos\frac{h\pi a}{a}dx$.

For example:- $f(x) = x$ in $0 < x < \pi, b < y < \pi$, (Note :- $a = \pi$).

$a_0 = \pi$.

$$a_n^* = \frac{2}{\pi n^2}[(-1)^n - 1], n = 1,2, \ldots.$$

Hence $u(x, y) = \frac{1}{2}(\pi - y) + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[(-1)^n - 1]\frac{\sinh n(\pi - y)}{\sinh n\pi}\cosh x$.

Example 2:

Solve $u_{tt} + a^2 u_{xxxx} = 0, 0 < x < l, t > 0$

$u(x, 0) = f(x), 0 \leq x \leq l \ldots$

$u_t(x, 0) = g(x), 0 \leq x \leq l.$

$u(0, t) = u(l, t) = 0, t > 0.$

$u_{xx}(0, t) = u_{xx}(l, t) = 0.$

Solution:

Assume a nontrivial solution in the form,

Now $u(x, t) = x(x)T(t)$.

$$u_{tt} + a^2 u_{x\times xx} = 0$$
$$xT'' + a^2 x^{(1r)} T = 0$$
$$xT'' = -a^2 x^{(1v)} T.$$
$$-\frac{1}{a^2}\frac{T''}{T} = \frac{x^{(1v)}}{x}.$$

Differentiating $\frac{d}{dt}\left(\frac{T''}{T}\right) \cdot 0.$

Integrating, we get, $\frac{1}{a^2}\frac{T''}{T} = \frac{x^{(1/v)}}{x} = \alpha^4.$

$$\frac{-T''}{a^2 T} = \alpha^4$$
$$\Rightarrow -T'' = \alpha^4 a^2 T$$
$$\Rightarrow T'' + \alpha^4 a^2 T = 0, \alpha > 0$$

$$\left[\begin{array}{c} \frac{x^{(1v)}}{x} = \alpha^4. \\ \Rightarrow x^{(q)} = \alpha^4 x \end{array} \right]$$
$$\Rightarrow x^{(12)} - \alpha^4 x = 0.$$

The equation of $x(x)$ has the given solution,

$x^{(n)} - \alpha^4 x = 0.$
Wax. , $D^4 - \alpha^4 = 0.$
$\phi^4 = \alpha^4.$
$D^2 = \pm\alpha^2.$
$\Rightarrow \phi^2 = \alpha^2.$
$D^2 = -\alpha^2$
$D = \pm\alpha$
$D = \pm\sqrt{-\alpha^2}$
$= ae^{\alpha x} + 6e^{-\alpha x}$
$= \pm\alpha i.$
$e^{\alpha x} = \cosh\alpha x + \sinh\alpha x.$
$e^{-\alpha x} = \cosh\alpha x - \sinh\alpha x$
$D = c\cos\alpha x + D\sin\alpha x.$
$\Rightarrow a(\cosh\alpha x + \sinh\alpha x) +$
$b(\cosh\alpha x - \sinh\alpha x)$
$\Rightarrow a\cosh\alpha x + a\sinh\alpha x +$
$b\cosh\alpha x - b\sinh\alpha x$
$\Rightarrow (a+b)\cosh\alpha x + (a-b)\sinh\alpha x$

$\Rightarrow A\cosh\alpha x + B\sinh\alpha x$ where $A = a+b, B = a-b$

The given solution of $x(x)$ is given by.

$x(x) = A\cosh\alpha x + B\sinh\alpha x + C\cos\alpha x + D\sin\alpha x.$
boundary and require that.

The boundary and require that;

$u(0,t) = x(0)T(t) = 0.$
$u(l,t) = x(d)T(t) = 0.$
$\Rightarrow x(0) = x(l) = 0$ and
$u_{xx}(0,t) = x''(0)T(t) = 0.$
$u_{xx}(l,t) = x''(l)T(t) = 0$
$\Rightarrow x''(0) = x''(l) = 0\ [\because T(t) \neq 0].$

Differentiation ' x ' twice w.r.to ' x ',

$x(x) = A\cosh\alpha x + B\sinh\alpha x + C\cos\alpha x + D\sin\alpha x.$
$x'(x) = A\alpha\sinh\alpha x + B\alpha\cosh\alpha x - C\alpha\sin\alpha x + D\alpha\cos\alpha x)$
$x''(x) = A\alpha^2\cosh\alpha x + B\alpha^2\sinh\alpha x - C\alpha^2\cos\alpha x - D\alpha^2\sin\alpha x$

Now applying the boundary condition,

$x(0) = x''(0) = 0$ yields.

$x(0) \Rightarrow A + C = 0$

$x''(0) \Rightarrow A\alpha^2 - c\alpha^2 = 0$

$\Rightarrow \alpha^2(A \leftarrow C) = 0.$

$\Rightarrow A - C = 0.$

$\Rightarrow A = c = 0 \ (\because \alpha > 0).$

Now, $x(l) = x''(l) = 0$ yields.

$x(l) = B\sinh \alpha l + D\sin \alpha l = 0$

$x''(l) = B\alpha^2\sinh \alpha l - D\alpha^2\sin \alpha l = 0 \ (\because A = C = 0)$

These equations are satisfied if $B \sinh \alpha l = 0, D \sin \alpha l = 0.$ since $\sinh \alpha l \neq 0, B = 0 \Rightarrow B$ must vanish for trivial solution, $\sin \alpha l = 0$ then $D \neq 0.$

$\alpha l = n\pi$

$\alpha = \frac{n\pi}{l} \ (\because \sin n\pi = 0)$

Hence $\alpha = \frac{m\pi}{l}; n = 1,2, ...$

we obtain,

$$x_n(x) = D_n \sin \frac{n\pi}{l} x \quad [\because A = B = c = 0 \text{ and } \alpha = \frac{n\pi}{l}].$$

The given solution for $T(t)$ is,

$\mathrm{T} + a^2\alpha^4 T = 0$

$D^2 + a^2\alpha^{(4)} = 0$

$D^2 = -a^2\alpha^4$

$D = \pm\sqrt{-a^2\alpha^4}$

$D = \pm a\alpha^2 i$

$\therefore T(t) = E\cos a\alpha^2 t + f\sin a\alpha^2 t.$

Inserting the values of α,

$$T_n(t) = E_n \cos a \left(\frac{n\pi}{l}\right)^2 t + F_n \sin a \left(\frac{n\pi}{l}\right)^2 t$$

Thus, the given solution of the equation for the transverse vibration of a beam is,

$$u(x,t) = \int_{n=1}^{\infty} D_n \sin \frac{n\pi x}{l} \left[E_n \cos a \left(\frac{n\pi}{l}\right)^2 t + F_n \sin a \left(\frac{n\pi}{l}\right)^2 t\right].$$

$$= D_n E_n \sin\frac{n\pi x}{l}\cos a\left(\frac{n\pi}{l}\right)^2 t + D_n F_n \sin\frac{n\pi x}{l}\sin a\left(\frac{n\pi}{l}\right)^2$$

$$= a_n \sin\frac{n\pi x}{l}\cos a\left(\frac{n\pi}{l}\right)^2 t + b_n \sin\frac{n\pi x}{l}\sin a\left(\frac{n\pi}{l}\right)^2 t.$$

$$= \sum_{n=1}^{\infty}\left[a_n \cos a\left(\frac{n\pi}{l}\right)^2 t + b_n \sin a\left(\frac{n\pi}{l}\right)^2 t\right]\sin\frac{n\pi x}{l}.$$

where $a_n = D_n E_n; b_n = D_n F_n$

To satisfy the initial condition, $u(x,0) = f(x)$. we must have.

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin^2\frac{n\pi x}{l}.$$

$$\therefore a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

Now, the application of 2nd initial condition gives

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos a\left(\frac{n\pi}{l}\right)^2 t + b_n \sin a\left(\frac{n\pi}{l}\right)^2 t\right]\sin\frac{n\pi x}{l}.$$

$$u_f(x,t) = \sum_{n=1}^{\infty}\left[-\left(\frac{n\pi}{l}\right)^2 a_n \sin a\left(\frac{n\pi}{l}\right)^2 t + b_n a\left(\frac{n\pi}{l}\right)^2 \cos a\left(\frac{n}{l}\right.\right.$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n a\left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l}$$

$$\therefore b_n = \frac{2}{al}\left(\frac{l}{n\pi}\right)^2\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

Thus, the solution of initial boundary value problem, is given by.

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos a\left(\frac{n\pi}{l}\right)^2 t + b_n' \sin a\left(\frac{n\pi}{l}\right)^2 t\right]\sin\frac{n\pi x}{l}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \text{ and.}$$

$$b_n = \frac{2}{al}\left[\frac{l}{n\pi}\right]^2\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

Exercises:

1.Solving the following initial-boundary value problems:

$$u_{tt} = c^2 u_{xx}, 0 < x < 1, t > 0$$
$$u(x,0) = x(1-x), 0 \le x \le 1$$
$$u_t(x,0) = 0 \ ,$$
$$u(0,t) = u(1,t) = 0$$

2.Determine the solutions of the following initial-boundary value problems:

(i)
$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < \pi, t > 0$$
$$u(x,0) = 0$$
$$u_t(x,0) = 8\, sin^2 x \ , 0 \le x \le \pi.$$
$$u(0,t) = u(\pi,t) = 0, t > 0.$$

(ii)
$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < 1, t > 0$$
$$u(x,0) = 0$$
$$u_t(x,0) = \mathrm{x}\sin x \ , 0 \le x \le \pi.$$
$$u(0,t) = u(1,t) = 0, t > 0.$$

Unit IV

Boundary Value Problems: Boundary value problems – Maximum and minimum principles – Uniqueness and continuity theorem – Dirichlet Problem for a circle, a circular annulus, a rectangle – Dirichlet problem involving Poisson equation – Neumann problem for a circle and a rectangle.

Chapter 4: Sections 4.1 to 4.9

4.1. Boundary-Value Problems:

In this chapter, we shall be concerned with boundary-value problems. Mathematically, a boundary-value problem is finding a function which satisfies a given partial differential equation and particular boundary conditions. Physically speaking, the problem is independent of time, involving only space coordinates. Just as initial value problems are associated with hyperbolic partial differential equations, boundary-value problems are associated with partial differential equations of elliptic type. In marked contrast to initial-value problems, boundary-value problems are considerably more difficult to solve. This is due to the physical requirement that solutions must attain in the large unlike the case of initial-value problems, where solutions in the small, say over a short interval of time, may still be of physical interest.

The second-order partial differential equation of the elliptic type in n independent variables $x_1, x_2, \dots, x_n$ is of the form $\nabla^2 u = F\left(x_1, x_2, \dots, x_n, u_{x_1}, u_{x_2}, \dots, u_{x_n}\right)$ …….(1)

Some well-known elliptic equations include

A. Laplace equation: $\nabla^2 u = 0$ ……(2)

B. Poisson equation: $\nabla^2 u = g(x)$ ………(3)

$\nabla^2 u = \sum_{i=1}^{n} u_{x_i x_i}$ Where $g(x) = g(x_1, x_2, \dots, x_n)$

C. Helmholtz equation: $\nabla^2 u + \lambda u = 0$ ………(4)

where λ is a positive constant.

D. Schrodinger equation (time independent)

$\nabla^2 u + [\lambda - q(x)]u = 0$ ………… (5)

Let us first define a harmonic function. A function is said to be harmonic in a domain D if it satisfies the Laplace equation and if it and its first two derivatives are continuous in D.

Since the Laplace equation is linear and homogeneous, a linear combination of harmonic functions is harmonic.

1. The First Boundary-Value Problem

(The Dirichlet Problem): Find a function $u(x, y)$, harmonic in D, which satisfies

$u = f(s)$ on B ………. (6)

where $f(s)$ is a prescribed continuous function on the boundary B of the domain D. D is the interior of a simple closed piecewise smooth curve B.

We may physically interpret the solution u of the Dirichlet problem as the steady-state temperature distribution in a body containing no sources or sinks of heat, with the temperature prescribed at all points on the boundary.

2. The Second Boundary-Value Problem

(The Neumann Problem): Find a function $u(x, y)$, harmonic in D, which satisfies

$\frac{\partial u}{\partial n} = f(s)$ on B ……… (7)

With $\int_B f(s)ds = 0$ ……….(8)

The symbol $\partial u/\partial n$ denotes the directional derivative of u along the outward normal to the boundary B. The last condition (8) is known as the compatibility condition, since it is a consequence of (7) and the equation $\nabla^2 u = 0$. Here the solution u may be interpreted as the steady-state temperature distribution in a body containing no heat sources or heat sinks when the heat flux across the boundary is prescribed.

The compatibility condition, in this case, may be interpreted physically as the heat requirement that the net heat flux across the boundary be zero.

3. The Third Boundary-Value Problem

Find a function $u(x, y)$ harmonic in D which satisfies $\frac{\partial u}{\partial n} + h(s)u = f(s)$ on B ………(9)

where h and f are given continuous functions. In this problem, the solution u may be interpreted as the steady-state temperature distribution in a body, from the boundary of which the heat radiates freely into the surrounding medium of prescribed temperature.

4. The Fourth Boundary-Value Problem

(The Robin Problem): Find a function $u(x, y)$, harmonic in D, which satisfies boundary conditions of different types on different portions of the boundary B. An example involving such boundary conditions is $u = f_1(s)$ on B_1 where $B = B_1 + B_2$.

Problems 1 through (4) are called interior boundary-value problems. These differ from exterior boundary-value problems in two respects:

i. For problems of the latter variety, part of the boundary is at infinity.

ii. Solutions of exterior problems must satisfy an additional requirement, namely, that of boundedness at infinity.

4.2. Maximum and Minimum Principles:

Theorem 1: (The Maximum Principle)

Suppose that $u(x, y)$ is harmonic in a bounded domain D and continuous in D $= D + B$. Then u attains its maximum on the boundary B of D.

Physically, we may interpret this as meaning that the temperature of a body which has neither a source nor a sink of heat acquires its largest (and smallest) values on the surface of the body, and the electrostatic potential in a region which does not contain any free charge attains its maximum (and minimum) values on the boundary of the region.

Proof:

Let the maximum of u on B be M. Let us now suppose that the maximum of u in D is not attained at any point of B. Then it must be attained at some point $P_0(x_0, y_0)$ in D. If $M_0 = u(x_0, y_0)$ denotes the maximum of u in D, then M_0 must also be the maximum of u in D.

Consider the function $v(x, y) = u(x, y) + \frac{M_0 - M}{4R^2}[(x - x_0)^2 + (y - y_0)^2]$ ……..(1)

where the point $P(x, y)$ is in D and where R is the radius of a circle containing D. Note that $v(x_0, y_0) = u(x_0, y_0) = M_0$

We have $v(x, y) \leqslant M + (M_0 - M)/2 = \frac{1}{2}(M + M_0) < M_0$ on B. Thus, $v(x, y)$ like $u(x, y)$ must attain its maximum at a point in D. It follows from the definition of v that

$v_{xx} + v_{yy} = u_{xx} + u_{yy} + \frac{(M_0 - M)}{R^2} = \frac{(M_0 - M)}{R^2} > 0$ ………. (2)

But for v to be a maximum in D,

$v_{xx} \leqslant 0,\ v_{yy} \leqslant 0$

Thus, $v_{xx} + v_{yy} \leqslant 0$

which contradicts Eq. (2). Hence the maximum of u must be attained on B.

Theorem 2 (The Minimum Principle):

If $u(x, y)$ is harmonic in a bounded domain D and continuous in D $= D + B$, then u attains its minimum on the boundary B of D.

Proof:

The proof follows directly by applying the preceding theorem to the harmonic function $-u(x, y)$.

As a result of the above theorems, we see that $u =$ constant which is evidently harmonic attains the same value in the domain D as on the boundary B.

4.3. Uniqueness and Continuity Theorems:

Theorem 1: (Uniqueness Theorem)

The solution of the Dirichlet problem, if it exists, is unique.

Proof:

Let $u_1(x, y)$ and $u_2(x, y)$ be two solutions of the Dirichlet problem. Then u_1 and u_2 satisfy

$$\nabla^2 u_1 = 0,\ \nabla^2 u_2 = 0 \ \text{ in } D$$
$$u_1 = f,\ u_2 = f \ \text{ on } B$$

Since u_1 and u_2 are harmonic in $D, u_1 - u_2$ is also harmonic in D. But

$u_1 - u_2 = 0$ on B

By the maximum-minimum principles $u_1 - u_2 = 0$

at all interior points of D. Thus, we have $u_1 = u_2$

Therefore, the solution is unique.

Theorem 2: (Continuity Theorem)

The solution of the Dirichlet problem depends continuously on the boundary data.

Proof:

Let u_1 and u_2 be the solutions of $\begin{aligned} \nabla^2 u_1 &= 0 \quad && \text{in } D \\ u_1 &= f_1 && \text{on } B \end{aligned}$

And $\begin{aligned} \nabla^2 u_2 &= 0 \quad && \text{in } D \\ u_2 &= f_2 && \text{on } B \end{aligned}$

If $v = u_1 - u_2$, then v satisfies

$$\begin{aligned} \nabla^2 v &= 0 \quad && \text{in } D \\ v &= f_1 - f_2 && \text{on } B \end{aligned}$$

By the maximum and minimum principles, $f_1 - f_2$ attains the maximum and minimum of v on B. Thus, if $|f_1 - f_2| < \varepsilon$, then

$-\varepsilon < v_{\min} \leqslant v_{\max} < \varepsilon$ on B

Thus, at any interior points in D, we have $-\varepsilon < v_{\min} \leqslant v \leqslant v_{\max} < \varepsilon$

Therefore $|v| < \varepsilon$ in D. Hence $|u_1 - u_2| < \varepsilon$

Theorem 3:

Let $\{u_n\}$ be a sequence of functions harmonic in D and continuous in D. Let f_i be the values of u_i on B. If $\{u_n\}$ converges uniformly on B, then it converges uniformly in.

Proof: D

By hypothesis $\{f_n\}$ converges uniformly on B. Thus, for $\varepsilon > 0$, there exists an integer N such that everywhere on B

$|f_n - f_m| < \varepsilon$ for $n, m > N$

It follows from the continuity theorem that for all $n, m > N$

$|u_n - u_m| < \varepsilon$ in D, and hence the theorem is proved.

4.4. Dirichlet Problem for a Circle:

1. Interior Problem

We shall now establish the existence of the solution of the Dirichlet problem for a circle.

The Dirichlet problem is $\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,\ 0 \leqslant r < a$ …….(1)

$u(a, \theta) = f(\theta)$ ……….. (2)

By the method of separation of variables, we seek a solution in the form

$u(r, \theta) = R(r)\Theta(\theta)$ …………. (3)

Substitution of this in Equation (1) yields

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

Hence $\begin{aligned} r^2R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda\Theta = 0 \end{aligned}$ ……….(4)

Because of the periodicity conditions $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$ which ensure that the function Θ is single-valued, the case $\lambda < 0$ does not yield an acceptable solution. When $\lambda = 0$, we have $u(r, \theta) = (A + B\log r)(C\theta + D)$

Since $\log r \to -\infty$ as $r \to 0+$ (note that $r = 0$ is a singular point of Eq. (1)), B must vanish in order for u to be finite at $r = 0$. C must also vanish in order for u to be periodic with period 2π. Hence the solution for $\lambda = 0$ is $u =$ constant. When $\lambda > 0$, the solution of Equation (4) is

$\Theta(\theta) = A\cos\sqrt{\lambda}\theta + B\sin\sqrt{\lambda}\theta$

The periodicity conditions imply $\sqrt{\lambda} = n$ for $n = 1,2,3,\ldots$

Equation (3) is the Euler equation and therefore the general solution is

$R(r) = Cr^{\sqrt{\lambda}} + Dr^{-\sqrt{\lambda}}$

Since $r^{-\sqrt{\lambda}} \to \infty$ as $r \to 0, D$ must vanish for u to be continuous at $r = 0$. Thus, the solution is

$u(r,\theta) = Cr^{\sqrt{\lambda}}(A\cos\sqrt{\lambda}\theta + B\sin\sqrt{\lambda}\theta)$ for $\sqrt{\lambda} = 1,2,\ldots$

Hence the general solution of Eq. (8.4.1) may be written in the form

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n\cos n\theta + b_n\sin n\theta) \quad \ldots\ldots\ldots.. (5)$$

where the constant term $a_0/2$ represents the solution for $\lambda = 0$, and where a_n and b_n are constants. Letting $\rho = r/a$, we have

$$u(\rho,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^n(a_n\cos n\theta + b_n\sin n\theta) \quad \ldots\ldots\ldots.. (6)$$

Our next task is to show that $u(r,\theta)$ is harmonic in $0 \leqslant r < a$ and continuous in $0 \leqslant r \leqslant a$. We must also show that u satisfies the boundary condition (2).

We first assume that a_n and b_n are the Fourier coefficients of $f(\theta)$, that is,

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta \quad n = 0,1,2,3,\ldots \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta \quad n = 1,2,3,\ldots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (7)$$

Thus, from their very definitions, a_n and b_n are bounded, that is, there exists some number $M > 0$ such that $|a_0| < M,\ |a_n| < M,\ |b_n| < M,\ n = 1,2,3,\ldots$

Thus, if we consider the sequence of functions $\{u_n\}$ defined by

$$u_n(\rho,\theta) = \rho^n(a_n\cos n\theta + b_n\sin n\theta) \quad \ldots\ldots\ldots.. (8)$$

we see that

$|u_n| < 2\rho_0^n M,\ 0 \leqslant \rho \leqslant \rho_0 < 1$

Hence in any closed circular region, series (8.4.6) converges uniformly.

Next, differentiate u_n with respect to r. Then for $0 \leqslant \rho \leqslant \rho_0 < 1$

$$\left|\frac{\partial u_n}{\partial r}\right| = \left|\frac{n}{a}\rho^{n-1}(a_n\cos n\theta + b_n\sin n\theta)\right| < 2\frac{n}{a}\rho_0^{n-1}M$$

Thus, the series obtained by differentiating series (6) term by term with respect to r converges uniformly. In a similar manner, we can prove that the series obtained by twice differentiating series (6) term by term with respect to r and θ converge uniformly. Consequently,

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$
$$= \sum_{n=1}^{\infty} \frac{\rho^{n-2}}{a^2}(a_n \cos n\theta + b_n \sin n\theta)[n(n-1) + n - n^2]$$
$$= 0,\ 0 \leqslant \rho \leqslant \rho_0 < 1$$

Since each term of series (6) is a harmonic function, and since the series converges uniformly, $u(r,\theta)$ is harmonic at any interior point of the region $0 \leqslant \rho < 1$. It now remains to show that u satisfies the boundary data $f(\theta)$.

Substitution of the Fourier coefficients a_n and b_n into Eq. (8.4.6) yields

$$u(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta + \frac{1}{\pi}\sum_{n=1}^{\infty} \rho^n \int_0^{2\pi} f(\tau)$$
$$\times [\cos n\tau \cos n\theta + \sin n\tau \sin n\theta]d\tau$$

The interchange of summation and integration is permitted due to the uniform convergence of the series. For $0 \leqslant \rho \leqslant 1$

$$1 + 2\sum_{n=1}^{\infty} [\rho^n \cos n(\theta - \tau)] = 1 + \sum_{n=1}^{\infty} \left[\rho^n e^{in(\theta-\tau)} + \rho^n e^{-in(\theta-\tau)}\right]$$
$$= 1 + \frac{\rho e^{i(\theta-\tau)}}{1 - \rho e^{i(\theta-\tau)}} + \frac{\rho e^{-i(\theta-\tau)}}{1 - \rho e^{-i(\theta-\tau)}}$$
$$= \frac{1-\rho^2}{1 - \rho e^{i(\theta-\tau)} - \rho e^{-i(\theta-\tau)} + \rho^2}$$
$$= \frac{1-\rho^2}{1 - 2\rho\cos(\theta-\tau) + \rho^2}$$

Hence $u(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\rho^2}{1-2\rho\cos(\theta-\tau)+\rho^2} f(\tau)d\tau$ ………….. (10)

The integral on the right side of (10) is called the Poisson integral formula for a circle.

Now if $f(\theta) = 1$, then according to series (9), $u(r,\theta) = 1$ for $0 \leqslant \rho \leqslant 1$. Thus, Equation (10) gives

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\rho^2}{1-2\rho\cos(\theta-\tau)+\rho^2} d\tau$$

Hence,

$$f(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\rho^2}{1-2\rho\cos(\theta-\tau)+\rho^2} f(\theta)d\tau,\ 0 \leqslant \rho < 1$$

Therefore $u(\rho,\theta) - f(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-\rho^2)[f(\tau)-f(\theta)]}{1-2\rho\cos(\theta-\tau)+\rho^2} d\tau$ ……….(11)

Since $f(\theta)$ is uniformly continuous on $[0,2\pi]$, for given $\varepsilon > 0$, there exists a positive number $\delta(\varepsilon)$ such that $|\theta - \tau| < \delta$ implies $| f(\theta) - f(\tau) | < \varepsilon$. If $|\theta - \tau| \geqslant \delta$ so that $\theta - \tau \neq 2n\pi$ for $n = 0,1,2, ...$, then

$$\lim_{\rho \to 1-} \frac{1-\rho^2}{1 - 2\rho\cos(\theta - \tau) + \rho^2} = 0$$

In other words, there exists ρ_0 such that if $|\theta - \tau| \geqslant \delta$, then

$$\frac{1-\rho^2}{1 - 2\rho\cos(\theta - \tau) + \rho^2} < \varepsilon$$

for $0 \leqslant \rho \leqslant \rho_0 < 1$. Hence Equation (10) yields

$$\begin{aligned}|u(r,\theta) - f(\theta)| &\leqslant \frac{1}{2\pi}\int_{|\theta-\tau|\geqslant\delta}^{2\pi} \frac{(1-\rho^2)|f(\theta) - f(\tau)|}{1 - 2\rho\cos(\theta - \tau) + \rho^2} d\tau \\ &\quad + \frac{1}{2\pi}\int_{|\theta-\tau|<\delta}^{2\pi} \frac{(1-\rho^2)|f(\theta) - f(\tau)|}{1 - 2\rho\cos(\theta - \tau) + \rho^2} d\tau \\ &\leqslant \frac{1}{2\pi} 2\pi\varepsilon \left[2 \max_{0\leqslant\theta\leqslant2\pi} |f(\theta)|\right] + \frac{\varepsilon}{2\pi} \cdot 2\pi \\ &\leqslant \varepsilon \left[1 + 2\left(\max_{0\leqslant\theta\leqslant2\pi} |f(\theta)|\right)\right]\end{aligned}$$

which implies that

$$\lim_{\rho \to 1-} u(r,\theta) = f(\theta)$$

uniformly in θ. Therefore, we state the following theorem.

Theorem 1:

There exists one and only one harmonic function $u(r,\theta)$ which satisfies the continuous boundary data $f(\theta)$. This function is either given by

$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2-r^2}{a^2-2ar\cos(\theta-\tau)+r^2} f(\tau) d\tau$ .... (12)

(or) $u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{r^n}{a^n}(a_n \cos n\theta + b_n \sin n\theta)$ ....(13)

where a_n and b_n are the Fourier coefficients of $f(\theta)$.

For $\rho = 0$, the Poisson integral formula (10) becomes

$u(0,\theta) = u(0) = \frac{1}{2\pi}\int_0^{2\pi} f(\tau) d\tau$ .... (14)

Theorem 2: (Mean Value Theorem):

If u is harmonic in a circle, then the value of u at the center is equal to the mean value of u on the boundary of the circle.

Several comments are in order. First, the Continuity Theorem 2 for the Dirichlet problem for the Laplace equation is a special example of the general result that the Dirichlet problems for all elliptic equations are well-posed. Second, the formula (12) represents a unique continuous solution of the Laplace equation in $0 \leqslant r < a$ even when $f(\theta)$ is discontinuous. This means that for Laplace's equation, discontinuities in boundary conditions are smoothed out in the interior of the domain. This is a remarkable contrast to the linear hyperbolic equations where any discontinuity in the data propagates along the characteristics. Third, the integral solution (12) can be written as

$$u(r,\theta) = \int_{-\pi}^{\pi} P(r,\tau-\theta) f(\tau) d\tau$$

where $P(r,\tau-\theta)$ is called the Poisson kernel given by

$$P(r,\tau-\theta) = \frac{1}{2\pi} \frac{(a^2-r^2)}{[a^2 - 2ar\cos(\tau-\theta) + r^2]}$$

Clearly, $P(a,\tau-\theta) = 0$ except at $\tau = \theta$. Also

This implies that

$$f(\theta) = \lim_{r\to a^-} u(r,\theta) = \int_{-\pi}^{\pi} \lim_{r\to a^-} P(r,\tau-\theta) f(\tau) d\tau$$

$$\lim_{r\to a^-} P(r,\tau-\theta) = \delta(\tau-\theta)$$

where $\delta(x)$ is the Dirac delta function.

As in the preceding section, the exterior Dirichlet problem for a circle can readily be solved. For the exterior problem u must be bounded as $r \to \infty$. The general solution, therefore, is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r^{-n}}{a^{-n}}\right) (a_n \cos n\theta + b_n \sin n\theta) \quad \text{……….. (15)}$$

Applying the boundary condition $u(a,\theta) = f(\theta)$, we obtain

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

Hence, we find

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(\tau)\cos n\tau d\tau, \quad n = 0,1,2,\ldots \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(\tau)\sin n\tau d\tau, \quad n = 1,2,3,\ldots \end{aligned} \quad \text{…………(17)}$$

Substitution of a_n and b_n into Equation (15) yields

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \left[1 + 2\sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \cos n(\theta-\tau)\right] f(\tau) d\tau$$

Comparing with Equation (9), we see that the only difference between the exterior and interior problem is that ρ^n is replaced by ρ^{-n}. Therefore, the final result takes the form

$$u(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2-1}{1-2\rho\cos(\theta-\tau)+\rho^2} f(\tau)d\tau \quad \ldots\ldots\ldots\ldots (18) \text{ for } \rho > 1.$$

4.5. Dirichlet Problem for a Circular Annulus:

The natural extension of the Dirichlet problem for a circle is the Dirichlet problem for a circular annulus, that is

$\nabla^2 u = 0,\ r_2 < r < r_1$ ……….. (1)

$u(r_1,\theta) = f(\theta)$ ……….. (2)

$u(r_2,\theta) = g(\theta)$ …………. (3)

In addition $u(r,\theta)$ must satisfy the periodicity condition. Accordingly, $f(\theta)$ and $g(\theta)$ must also be periodic with period 2π.

Proceeding as in the case of the Dirichlet problem for a circle, we obtain for $\lambda = 0$

$$u(r,\theta) = (A + B\log r)(C\theta + D)$$

The periodicity condition on u requires that $C = 0$. Then $u(r,\theta)$ becomes

$$u(r,\theta) = \frac{a_0}{2} + \frac{b_0}{2}\log r$$

where $a_0 = 2AD$ and $b_0 = 2BD$.

The solution for the case $\lambda > 0$ is

$$u(r,\theta) = \left(Cr^{\sqrt{\lambda}} + Dr^{-\sqrt{\lambda}}\right)(A\cos\sqrt{\lambda}\theta + B\sin\sqrt{\lambda}\theta)$$

for $\sqrt{\lambda} = n = 1,2,3, \ldots$ Thus, the general solution is

$$u(r,\theta) = \frac{1}{2}(a_0 + b_0\log r) + \sum_{n=1}^{\infty} \left[(a_n r^n + b_n r^{-n})\cos n\theta + (c_n r^n + d_n r^{-n})\sin n\theta\right] \quad \ldots\ldots\ldots\ldots (4)$$

where a_n, b_n, c_n, and d_n are constants.

Applying the boundary conditions (2) and (3), we find that the coefficients are given by

$$a_0 + b_0\log r_1 = \frac{1}{\pi}\int_0^{2\pi} f(\tau)d\tau$$

$$a_n r_1^n + b_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f(\tau)\cos n\tau d\tau$$

$$c_n r_1^n + d_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f(\tau)\sin n\tau d\tau$$

and

$$a_0 + b_0 \log r_2 = \frac{1}{\pi}\int_0^{2\pi} g(\tau)d\tau$$
$$a_n r_2^n + b_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} g(\tau)\cos n\tau d\tau$$
$$c_n r_2^n + d_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} g(\tau)\sin n\tau d\tau$$

The constants $a_0, b_0, a_n, b_n, c_n, d_n$ for $n = 1,2,3, \ldots$ can then be determined. Hence the solution of the Dirichlet problem,

4.6. Dirichlet Problem for a Rectangle:

Let us first consider the problem

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \qquad 0 < x < a, 0 < y < b \quad \ldots\ldots\ldots\ldots(1)$$
$$u(x,0) = f(x), \qquad 0 \leqslant x \leqslant a \quad \ldots\ldots\ldots\ldots(2)$$
$$u(x,b) = 0 \quad \ldots\ldots\ldots\ldots(3)$$
$$u(0,y) = 0 \quad \ldots\ldots\ldots\ldots(4)$$
$$u(a,y) = 0 \quad \ldots\ldots\ldots(5)$$

We seek a solution in the form

$u(x,y) = X(x)Y(y)$

Substituting $u(x,y)$ in the Laplace equation, we obtain

$X'' - \lambda X = 0$ ………… (6)

$Y'' + \lambda Y = 0$ ………….()

where λ is a separation constant. Since the boundary conditions are homogeneous on $x = 0$ and $x = a$, we choose $\lambda = -\alpha^2$ with $\alpha > 0$ in order to obtain nontrivial solutions of the eigenvalue problem

$$X'' + \alpha^2 X = 0$$
$$X(0) = X(a) = 0$$

It is easily found that the eigenvalues are $\alpha = \frac{n\pi}{a}$, $n = 1,2,3, \ldots$

and the corresponding Eigen functions are $\sin n\pi x/a$. Hence

$$X_n(x) = B_n \sin \frac{n\pi x}{a}$$

The solution of Eq. (8.7.7) is $Y(y) = C\cosh \alpha y + D\sinh \alpha y$, which may also be written in the form $Y(y) = E\sinh \alpha(y + F)$

where $E = (D^2 - C^2)^{\frac{1}{2}}$ and $F = 1/\alpha \tanh^{-1}(C/D)$. Applying the remaining homogeneous boundary condition $u(x,b) = X(x)Y(b) = 0$

we obtain $Y(b) = E\sinh\alpha(b+F) = 0$

and hence $F = -b,\ E \neq 0$

for a nontrivial solution $u(x,y)$. Thus, we have $Y_n(y) = E_n\sinh\frac{n\pi}{a}(y-b)$

Because of linearity, the solution is $u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi}{a}(y-b)$

where $a_n = B_n E_n$. Now, we apply the nonhomogeneous boundary condition to obtain

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{-n\pi b}{a}\right)\sin\frac{n\pi x}{a}$$

This is a Fourier sine series and hence0

$$a_n = \frac{-2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

Thus, the formal solution is given by $u(x,y) = \sum_{n=1}^{\infty} a_n^* \frac{\sinh\frac{n\pi}{a}(b-y)}{\sinh\frac{n\pi b}{a}}\sin\frac{n\pi x}{a}$ ....(8)

Where $a_n^* = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a} d$

To prove the existence of solution (8.7.8), we first note that

$$\frac{\sinh\frac{n\pi}{a}(b-y)}{\sinh\frac{n\pi b}{a}} = e^{-n\pi y/a}\left[\frac{1-e^{-(2n\pi/a)(b-y)}}{1-e^{-2n\pi b/a}}\right]$$

$$\leqslant C_1 e^{-n\pi y/a}$$

where C_1 is a constant. Since $f(x)$ is bounded, we have

$$|a_n^*| \leqslant \frac{2}{a}\int_0^a |f(x)|dx = C_2$$

Thus, the series for $u(x,y)$ is dominated by the series

$$\sum_{n=1}^{\infty} Me^{-n\pi y_d aa} \text{ for } y \geqslant y_0 > 0,\ M = \text{ constant}$$

and hence $u(x,y)$ converges uniformly in x and y whenever $0 \leqslant x \leqslant a,\ y \geqslant y_0 > 0$. Consequently, $u(x,y)$ is continuous in this region and satisfies the boundary values $u(0,y) = u(a,y) = u(x,b) = 0$.

Now differentiating u twice with respect to x, we obtain

$$u_{xx}(x,y) = \sum_{n=1}^{\infty} -a_n^*\left(\frac{n\pi}{a}\right)^2 \frac{\sinh\frac{n\pi}{a}(b-y)}{\sinh\frac{n\pi b}{a}}\sin\frac{n\pi x}{a}$$

and differentiating u twice with respect to y, we obtain

$$u_{yy}(x,y)=\sum_{n=1}^{\infty} a_n^*\left(\frac{n\pi}{a}\right)^2 \frac{\sinh\frac{n\pi}{a}(b-y)}{\sinh\frac{n\pi b}{a}}\sin\frac{n\pi x}{a}$$

It is evident that the series for u_{xx} and u_{yy} are both dominated by

$$\sum_{n=1}^{\infty} M^* n^2 e^{-n\pi y_0/a}$$

and hence converge uniformly for any $0<y_0<b$. It follows that u_{xx} and u_{yy} exist, and u satisfies the Laplace equation.

It now remains to be shown that $u(x,0)=f(x)$. Let $f(x)$ be a continuous function and let $f'(x)$ be piecewise continuous on $[0,a]$. If, in addition, $f(0)=f(a)=0$, then the Fourier series for $f(x)$ converges uniformly. Putting $y=0$ in the series for $u(x,y)$, we obtain

$$u(x,0)=\sum_{n=1}^{\infty} a_n^*\sin\frac{n\pi x}{a}$$

Since $u(x,0)$ converges uniformly to $f(x)$, we write for $\varepsilon>0$

$|s_m(x,0)-s_n(x,0)|<\varepsilon$ for $m,n>N_\varepsilon$

Where $s_m(x,y)=\sum_{n=1}^{m} a_n^*\sin\frac{n\pi x}{a}$

We know that $s_m(x,y)-s_n(x,y)$ satisfies the Laplace equation and the boundary conditions on $x=0, x=a$ and $y=b$. Then by the maximum principle,

$|s_m(x,y)-s_n(x,y)|<\varepsilon$ for $m,n>N_\varepsilon$

in the region $0\leqslant x\leqslant a, 0\leqslant y\leqslant b$. Thus, the series for $u(x,y)$ converges uniformly, and as a consequence, $u(x,y)$ is continuous in the region $0\leqslant x\leqslant a, 0\leqslant y\leqslant b$. Hence, we obtain

$$u(x,0)=\sum_{n=1}^{\infty} a_n^*\sin\frac{n\pi x}{a}=f(x)$$

Thus the solution (8) is established.

The general Dirichlet problem

$$\nabla^2 u=0,\ 0<x<a,\ 0<y<b$$
$$u(x,0)=f_1(x)$$
$$u(x,a)=f_2(x)$$
$$u(0,y)=f_3(y)$$
$$u(b,y)=f_4(y)$$

can be solved by separating it into four problems, each of which has one nonhomogeneous boundary condition and the rest zero. Thus, determining each solution as in the preceding problem and then adding the four solutions, the solution of the Dirichlet problem for a rectangle is obtained.

4.7. Dirichlet Problem Involving Poisson Equation:

The solution of the Dirichlet problem involving the Poisson equation can be obtained for simple regions when the solution of the corresponding Dirichlet problem for the Laplace equation is known.

Consider the Poisson equation $\nabla^2 u = u_{xx} + u_{yy} = f(x, y)$ in D

with the condition $u = g(x, y)$ on B

Assume that the solution can be written in the form $u = v + w$

where v is a particular solution of the Poisson equation and w is the solution of the associated homogeneous equation, that is,
$$\nabla^2 v = f$$
$$\nabla^2 w = 0$$

As soon as v is ascertained, the solution of the Dirichlet problem

$$\begin{aligned} \nabla^2 w &= 0 && \text{in } D \\ w &= -v + g(x, y) && \text{on } B \end{aligned}$$

can be determined. The usual method of finding a particular solution for the case in which $f(x, y)$ is a polynomial of degree n is to seek a solution in the form of a polynomial of degree $(n + 2)$ with undetermined coefficients.

As an example, consider the torsion problem

$$\nabla^2 u = -2, \ 0 < x < a, \ 0 < y < b$$
$$u(0, y) = 0$$
$$u(a, y) = 0$$
$$u(x, 0) = 0$$
$$u(x, b) = 0$$

We let $u = v + w$. Now assume v to be of the form

$$v(x, y) = A + Bx + Cy + Dx^2 + Exy + Fy^2$$

Substituting this in the Poisson equation, we obtain

$$2D + 2F = -2$$

The simplest way of satisfying this equation is to choose

$D = -1$ and $F = 0$

The remaining coefficients are arbitrary. Thus we take

$$v(x,y) = ax - x^2$$

so that v reduces to zero on the sides $x = 0$ and $x = a$. Next, we find w from

$$\nabla^2 w = 0,\ 0 < x < a,\ 0 < y < b$$
$$w(0,y) = -v(0,y) = 0$$
$$w(a,y) = -v(a,y) = 0$$
$$w(x,0) = -v(x,0) = -(ax - x^2)$$
$$w(x,b) = -v(x,b) = -(ax - x^2)$$

As in the Dirichlet problem the solution is found to be

$$w(x,y) = \sum_{n=1}^{\infty} \left(a_n \cosh\frac{n\pi y}{a} + b_n \sinh\frac{n\pi y}{a}\right) \sin\frac{n\pi x}{a}$$

Application of the nonhomogeneous boundary conditions yield

$$w(x,0) = -(ax - x^2) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{a}$$

$$w(x,b) = -(ax - x^2) = \sum_{n=1}^{\infty} \left(a_n \cosh\frac{n\pi b}{a} + b_n \sinh\frac{n\pi b}{a}\right) \sin\frac{n\pi x}{a}$$

from which we find

$$a_n = \frac{2}{a}\int_0^a (x^2 - ax)\sin\frac{n\pi x}{a}\,dx$$
$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-8a^2}{\pi^3 n^3} & \text{if } n \text{ is odd} \end{cases}$$

and

$$\left(a_n \cosh\frac{n\pi b}{a} + b_n \sinh\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a (x^2 - ax)\sin\frac{n\pi x}{a}\,dx$$

Thus, we have

$$b_n = \frac{\left(1 - \cosh\frac{n\pi b}{a}\right) a_n}{\sinh\frac{n\pi b}{a}}$$

Hence the solution of the Dirichlet problem for the Poisson equation is given by

$$u(x,y) = (a - x)x$$
$$-\frac{8a^2}{\pi^3}\sum_{n=1}^{\infty} \frac{\left[\sinh(2n-1)\frac{\pi(b-y)}{a} + \sinh(2n-1)\frac{\pi y}{a}\right]}{\sinh(2n-1)\frac{\pi b}{a}} \frac{\sin(2n-1)\frac{\pi x}{a}}{(2n-1)^3}$$

4.8. Neumann Problem for a Circle:

Let u be a solution of the Neumann problem

$\nabla^2 u = 0$ in D

$\frac{\partial u}{\partial n} = f$ on B

It is evident that u+ constant is also a solution. Thus, we see that the solution of the Neumann problem is not unique, and it differs from another by a constant.

Consider the interior Neumann problem

$\nabla^2 u = 0$ r < R ………… (1)

$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} = f(\theta), r = R$ ……… (2)

Before we determine a solution of the Neumann problem, a necessary condition for the existence of a solution will be established.

In Green's second formula

$$\iint_D (\, v\, \nabla^2 u - u\, \nabla^2 v)\, ds = \int_B \left(v\, \frac{\partial u}{\partial n} - u\, \frac{\partial v}{\partial n}\right) ds \quad \dots\dots\dots\dots (3)$$

We put v=1, so that $\nabla^2 v = 0$ in D and $\frac{\partial v}{\partial n} = 0 \; on \; B$. Then, the result is

$\iint_D \nabla^2 u \, ds = \int_B \frac{\partial u}{\partial n} \, ds$ ………… (4)

Substituting of (1) and (2) into equation (4) yields

$\int_B \mathrm{f} \, ds = 0$ ………… (5)

Which may also be written in the form

$R \int_0^{2\pi} f(\theta) \, d\,\theta = 0$ ………….. (6)

As in the case of the interior Dirichlet problem for a circle, the solutions of the Laplace equation is $\mathrm{u}\,(r, \theta) = \frac{a_0}{2} \sum_{k=1}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta)$ …………..(7)

Differentiating this with respect to r and applying the boundary condition (2), we obtain

$\frac{\partial u}{\partial r}(R, \theta) = \sum_{k=1}^{\infty} kR^{k-1}(a_k \cos k\theta + b_k \sin k\theta) = f(\theta)$ ……….. (8)

Hence the coefficients are given by

$$a_k = \frac{1}{k\pi R^{k-1}} \int_0^{2\pi} f(\tau)\cos k\tau d\tau, \quad k = 1,2,3, \dots$$
$$b_k = \frac{1}{k\pi R^{k-1}} \int_0^{2\pi} f(\tau)\sin k\tau d\tau, \quad k = 1,2,3, \dots \quad \dots\dots\dots\dots\dots (9)$$

Note that the expansion of $f(\theta)$ in a series of the form (8.6.8) is possible only by virtue of the compatibility condition (8.6.6) since

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\tau)d\tau = 0$$

Inserting a_k and b_k in Equation (7), we obtain

$$u(r,\theta) = \frac{a_0}{2} + \frac{R}{\pi}\int_0^{2\pi} \left[\sum_{k=1}^{\infty} \left(\frac{r}{R}\right)^k \cos k(\theta - \tau)\right] f(\tau)d\tau$$

Using the identity

$$-\frac{1}{2}\log[1 + \rho^2 - 2\rho\cos(\theta - \tau)] = \sum_{k=1}^{\infty} \frac{1}{k}\rho^k \cos k(\theta - \tau)$$

with $\rho = r/R$, we find that

$$u(r,\theta) = \frac{a_0}{2} - \frac{R}{2\pi}\int_0^{2\pi} \log[R^2 - 2rR\cos(\theta - \tau) + r^2] f(\tau)d\tau \quad \dots\dots\dots\dots (10)$$

in which a constant factor R^2 in the argument of the logarithm was eliminated by virtue of Equation (6).

In a similar manner, for the exterior Neumann problem, we can readily find that

$$u(r,\theta) = \frac{a_0}{2} + \frac{R}{2\pi}\int_0^{2\pi} \log[R^2 - 2rR\cos(\theta - \tau) + r^2] f(\tau)d\tau \quad \dots\dots\dots\dots (11)$$

4.9. The Neumann Problem for a Rectangle:

Consider the Neumann problem

$\nabla^2 u = 0, \ 0 < x < a, \ 0 < y < b$ ……. (1)

$u_x(0,y) = f_1(y)$ ……….. (2)

$u_x(a,y) = f_2(y)$ ………… (3)

$u_y(x,0) = g_1(x)$ ………. (4)

$u_y(x,b) = g_2(x)$ ………… (5)

The compatibility condition that must be fulfilled in this case is

$\int_0^a [g_1(x) - g_2(x)]dx + \int_0^b [f_1(y) - f_2(y)]dy = 0$ ……….. (6)

We assume a solution in the form $u(x,y) = u_1(x,y) + u_2(x,y)$ ………..(7)

where $u_1(x,y)$ is a solution of

$$\nabla^2 u_1 = 0$$
$$\frac{\partial u_1}{\partial x}(0, y) = 0$$
$$\frac{\partial u_1}{\partial y}(x, 0) = g_1(x)$$
$$\frac{\partial u_1}{\partial y}(x, b) = g_2(x)$$

and where g_1 and g_2 satisfy the compatibility condition

$\int_0^a [g_1(x) - g_2(x)]dx = 0$ ……….. (9)

The function $u_2(x, y)$ is a solution of

$$\nabla^2 u_2 = 0$$
$$\frac{\partial u_2}{\partial x}(0, y) = f_1(y)$$
$$\frac{\partial u_2}{\partial y}(x, 0) = 0$$
$$\frac{\partial u_2}{\partial y}(x, b) = 0$$

where f_1 and f_2 satisfy the compatibility condition

$\int_0^b [f_1(y) - f_2(y)]dy = 0$ ………… (11)

$u_1(x, y)$ and $u_2(x, y)$ can be determined. Conditions (9) and (11) ensure that condition (6) is fulfilled. Thus the problem is solved.

However, the solution obtained in this manner is rather restrictive. In general, condition (6) does not imply conditions (9) and (11). Thus, generally speaking, it is not possible to obtain a solution of the Neumann problem for a rectangle by the method described above.

Assume that, a solution in the form $u(x, y) = \frac{Y_0}{2}(y) + \sum_{n=1}^{\infty} X_n(x)Y_n(y)$ …………(12)

where $X_n(x) = \cos n\pi x/a$ is an Eigen function of the eigenvalue problem

$$X'' + \lambda X = 0$$
$$X'(0) = X'(a) = 0$$

corresponding to the eigenvalue $\lambda_n = (n\pi/a)^2$. Then from Equation (12), we see that

$$Y_n(y) = \frac{2}{a}\int_0^a u(x, y)X_n(x)dx$$

Multiplying both sides of Equation (1) by $2\cos(n\pi x/a)$ and integrating with respect to x from 0 to a, we obtain $\frac{2}{a}\int_0^a (u_{xx} + u_{yy})\cos\frac{n\pi x}{a}dx = 0$

or $Y_n'' + \frac{2}{a}\int_0^a u_{xx}\cos\frac{n\pi x}{a}dx = 0$

Integrating the second term by parts and applying the boundary conditions (2) and (3), we obtain $Y_n'' - \left(\frac{n\pi}{a}\right)^2 Y_n = F_n(y)$ …………(14)

where $F_n(y) = 2[f_1(y) - (-1)^n f_2(y)]/a$. This is an ordinary differential equation whose solution may be written in the form

$$Y_n(y) = A_n \cosh\frac{n\pi y}{a} + B_n \sinh\frac{n\pi y}{a} + \frac{2}{\pi n}\int_0^y F_n(\tau)\sinh\frac{n\pi}{a}(y-\tau)d\tau \quad ……….. (15)$$

The coefficients A_n and B_n are determined from the boundary conditions

$$Y_n'(0) = \frac{2}{a}\int_0^a u_y(x,0)\cos\frac{n\pi x}{a}dx$$

And $Y_n'(b) = \frac{2}{a}\int_0^a g_2(x)\cos\frac{n\pi x}{a}dx$ ………….(17)

For $n = 0$, Equation (14) takes the form

$$Y_0'' = \frac{2}{a}[f_1(y) - f_2(y)]$$

and hence

$$Y_0' = \frac{2}{a}\int_0^y [f_1(\tau) - f_2(\tau)]d\tau + C$$

where C is an integration constant. Employing the condition (16) for $n = 0$, we find

$$C = \frac{2}{a}\int_0^a g_1(x)dx$$

Thus, we have

$$Y_0'(y) = \frac{2}{a}\left\{\int_0^y [f_1(\tau) - f_2(\tau)]d\tau + \int_0^a g_1(x)dx\right\}$$

Consequently,

$$Y_0'(b) = \frac{2}{a}\left\{\int_0^b [f_1(\tau) - f_2(\tau)]d\tau + \int_0^a g_1(x)dx\right\}$$

Also from Equation (16), we have

$$Y_0'(b) = \frac{2}{a}\int_0^a g_2(x)dx$$

It follows from these two expressions for $Y_0'(b)$ that

$$\int_0^b [f_1(y) - f_2(y)]dy + \int_0^a [g_1(x) - g_2(x)]dx = 0$$

which is the necessary condition for the existence of a solution to the Neumann problem for a rectangle.

Exercises:

1.Reduce the Neumann problem to the Dirichlet problem in the two dimensional case.

2.Reduce the wave equation $u_{\mathrm{n}} = c^2(u_{\mathrm{xx}} + u_{\mathrm{yy}} + u_{\mathrm{zz}})$

to the Laplace equation $u_{\mathrm{xx}} + u_{\mathrm{yy}} + u_{\mathrm{zz}} + u_{\tau\tau} = 0$, by letting $\tau = ict$ where $i = \sqrt{-1}$. Obtain the solution of the wave equation in cylindrical coordinates via the solution of the Laplace equation. Assume that $u(r, \theta, z, \tau)$ is independent of z.

3.Prove that a function which is harmonic everywhere on a plane and is bounded either above or below is a constant. This is called the Liouvile theorem.

Unit V

The Delta function – Green's function – Method of Green's function – Dirichlet Problem for the Laplace and Helmholtz operators – Method of images and Eigen functions – Higher dimensional problem – Neumann Problem.

Chapter 5: Section 5.1 to 5.9

5.1. The Delta Function:

The Green's function method is applied here to boundary-value problems in partial differential equations. The method provides solutions in integral form and is applicable to a wide class of problems in applied mathematics and mathematical physics.

Before developing the method of Green's function, we will first define the Dirac delta function $\delta(x-\xi, y-\eta)$ in two dimensions by

a. $\delta(x-\xi, y-\eta) = 0,\ x \neq \xi,\ y \neq \eta$ ……..(1)

b. $\iint_{R_\varepsilon} \delta(x-\xi, y-\eta)dxdy = 1,\ R_\varepsilon: (x-\xi)^2 + (y-\eta)^2 < \varepsilon^2$ ……….(2)

c. $\iint_R F(x,y)\delta(x-\xi, y-\eta)dxdy = F(\xi,\eta)$ ………..(3)

for arbitrary continuous function F in the region R.

The delta function is not a function in the ordinary sense. It is a symbolic function, and is often viewed as the limit of a distribution.

If $\delta(x-\xi)$ and $\delta(y-\eta)$ are one-dimensional delta functions, we have

$\iint_R F(x,y)\delta(x-\xi)\delta(y-\eta)dxdy = F(\xi,\eta)$ ……….. (4)

Since (3) and (4) hold for an arbitrary continuous function F, we conclude that

$\delta(x-\xi, y-\eta) = \delta(x-\xi)\delta(y-\eta)$ …………. (5)

Thus, we may state that the two-dimensional delta function is the product of one-dimensional delta functions.

Higher dimensional delta functions can be defined in a similar manner.

5.2. Green's Function:

The solution of the Dirichlet problem

is given by $\begin{aligned} \nabla^2 u &= h(x,y) && \text{in } D \\ u &= f(x,y) && \text{on } B \end{aligned}$ ……..(1)

$u(x,y) = \iint_D G(x,y;\xi,\eta)h(\xi,\eta)d\xi d\eta + \int_B f\frac{\partial G}{\partial n}ds$ ………… (2)

where G is the Green's function and n denotes the outward normal to the boundary B of the region D. It is rather obvious then that the solution $u(x,y)$ can be determined as soon as the Green's function G is ascertained, so the problem in this technique is really to find the Green's function.

First we shall define the Green's function for the Dirichlet problem involving the Laplace operator. Then, the Green's function for the Dirichlet problem involving the Helmholtz operator may be defined in a completely analogous manner.

The Green's function for the Dirichlet problem involving the Laplace operator is the function which satisfies

a. $\nabla^2 G = \delta(x-\xi, y-\eta)^{12}$ in D …………(3)

$G = 0$ on B …………(4)

b. G is symmetric, that is, $G(x,y;\xi,\eta) = G(\xi,\eta;x,y)$ …………(5)

c. G is continuous in x, y, ξ, η, but $\partial G/\partial n$ has a discontinuity at the point (ξ,η) which is specified by the equation

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{\partial G}{\partial n} ds = 1 \quad \text{……..} (6)$$

where n is the outward normal to the circle

$C_\varepsilon: (x-\xi)^2 + (y-\eta)^2 = \varepsilon^2$

The Green's function G may be interpreted as the response of the system at a field point (x,y) due to a δ function input at the source point (ξ,η). G is continuous everywhere in D, and its first and second derivatives are continuous in D except at (ξ,η). Thus, property (a) essentially states that $\nabla^2 G = 0$ everywhere except at the source point (ξ,η).

We will now prove property (b).

Theorem 1:

The Green's function is symmetric.

Proof:

Applying Green's second formula

$$\iint_D (\phi\nabla^2\psi - \psi\nabla^2\phi) dS = \int_B \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) ds \quad \text{………..} (7)$$

to the functions $\phi = G(x,y;\xi,\eta)$ and $\psi = G(x,y;\xi^*,\eta^*)$ we obtain

$$\iint_D [G(x,y;\xi,\eta)\nabla^2 G(x,y;\xi^*,\eta^*) - G(x,y;\xi^*,\eta^*)\nabla^2 G(x,y;\xi,\eta)]dxdy$$
$$= \int_B \left[G(x,y;\xi,\eta)\frac{\partial G}{\partial n}(x,y;\xi^*,\eta^*) - G(x,y;\xi^*,\eta^*)\frac{\partial G}{\partial n}(x,y;\xi,\eta)\right] ds$$

Since $G(x,y;\xi,\eta)$ and hence $G(x,y;\xi^*,\eta^*)$ must vanish on B, we have

$$\iint_D [G(x,y;\xi,\eta)\nabla^2 G(x,y;\xi^*,\eta^*) - G(x,y;\xi^*,\eta^*)\nabla^2 G(x,y;\xi,\eta)]dxdy = 0$$

But $\nabla^2 G(x,y;\xi,\eta) = \delta(x-\xi, y-\eta)$

And $\nabla^2 G(x,y;\xi^*\eta^*) = \delta(x-\xi^*, y-\eta^*)$

Since $\iint_D G(x,y;\xi,\eta)\delta(x-\xi^*, y-\eta^*)dxdy = G(\xi^*,\eta^*;\xi,\eta)$and

$$\iint_D G(x,y;\xi^*,\eta^*)\delta(x-\xi, y-\eta)dxdy = G(\xi,\eta;\xi^*,\eta^*)$$

we obtain $G(\xi,\eta;\xi^*,\eta^*) = G(\xi^*,\eta^*;\xi,\eta)$

Theorem 2:

$\partial G/\partial n$ is discontinuous at (ξ,η) in particular,

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{\partial G}{\partial n}ds = 1,\ C_\varepsilon: (x-\xi)^2 + (y-\eta)^2 = \varepsilon^2$$

Proof:

Let R_ε be the region bounded by C_ε. Then integrating both sides of Equation (3), we obtain

$$\iint_{R_\varepsilon}\nabla^2 G dxdy = \iint_R \delta(x-\xi, y-\eta)dxdy = 1$$

It therefore follows that

$\lim_{\varepsilon\to 0}\iint_{R_\varepsilon}\nabla^2 G dxdy = 1$ ……….. (8)

Thus, by the Divergence theorem,

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{\partial G}{\partial n}ds = 1$$

5.3. Method of Green's Function:

It is often convenient to seek G as the sum of a particular integral of the nonhomogeneous equation and the solution of the associated homogeneous equation. That is, G may assume the form $G(\xi,\eta;x,y) = F(\xi,\eta;x,y) + g(\xi,\eta;x,y)$ ………….(1)

where F, known as the free-space Green's function, satisfies

$\nabla^2 F = \delta(\xi - x, \eta - y)$ in D ………… (2)

and g satisfies $\nabla^2 g = 0$ in D …………..(3)

so that by superposition $G = F + g$. Also $G = 0$ on B requires that

$g = -F$ on B …………. (4)

Note that F need not satisfy the boundary condition.

Before we determine the solution of a particular problem, let us first find F for the Laplace and Helmholtz operators.

(1) Laplace Operator

In this case F must satisfy

$\nabla^2 F = \delta(\xi - x, \eta - y)$ in D

Then for $r = [(\xi - x)^2 + (\eta - y)^2]^{\frac{1}{2}} > 0$, that is, for $\xi \neq x, \eta \neq y$, we have by taking (x, y) as the center

$$\nabla^2 F = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) = 0$$

since F is independent of θ. The solution, therefore, is

$F = A + B\log r$

Applying condition (6), we see that

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} \frac{\partial F}{\partial n} ds = \lim_{\varepsilon \to 0} \int_0^{2\pi} \frac{B}{r} r d\theta = 1$$ [15]

Thus $B = 1/2\pi$ and A is arbitrary. For simplicity we choose $A = 0$. Then F takes the form

$F = \frac{1}{2\pi}\log r$ …………. (5)

(2) Helmholtz Operator

Here F is required to satisfy $\nabla^2 F + \kappa^2 F = \delta(x - \xi, y - \eta)$

Again for $r > 0$, we find

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \kappa^2 F = 0$$

Or $r^2 F_{rr} + rF_r + \kappa^2 r^2 F = 0$

This is the Bessel equation of order zero, the solution of which is

$F(\kappa r) = AJ_0(\kappa r) + BY_0(\kappa r)$

Since the behavior of J_0 at $r = 0$ is not singular, we set $A = 0$. Thus, we have

$$F(\kappa r) = BY_0(\kappa r)$$

But for very small r,

$$Y_0(\kappa r) \sim \frac{2}{\pi}\log r$$

Applying condition sec 5.2 equation(6), we obtain

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{\partial F}{\partial n}ds = \lim_{\varepsilon\to 0}\int_{C_\varepsilon} B\frac{\partial Y_0}{\partial r}ds = 1$$

and hence $B = 1/4$. Thus $F(\kappa r)$ becomes

$$F(\kappa r) = \frac{1}{4}Y_0(\kappa r) \qquad \ldots\ldots\ldots\ldots.. (6)$$

We may point out that, since

$(\nabla^2 + \kappa^2)$ approaches ∇^2 as $\kappa \to 0$

it should (and does) follow that

$$\frac{1}{4}Y_0(\kappa r) \to \frac{1}{2\pi}\log r \text{ as } \kappa \to 0+$$

5.4. Dirichlet Problem for the Laplace Operator:

We are now in a position to determine the solution of the Dirichlet problem

$$\begin{aligned} \nabla^2 u &= h \quad \text{in } D \\ u &= f \quad \text{on } B \end{aligned} \qquad \ldots\ldots\ldots.. (1)$$

by the method of Green's function.

By putting $\phi(\xi,\eta) = G(\xi,\eta;x,y)$ and $\psi(\xi,\eta) = u(\xi,\eta)$ in Equation(7), we obtain

$$\iint_D [G(\xi,\eta;x,y)\nabla^2 u - u(\xi,\eta)\nabla^2 G]d\xi d\eta = \int_B \left[G(\xi,\eta;x,y)\frac{\partial u}{\partial n} - u(\xi,\eta)\frac{\partial G}{\partial n}\right]ds$$

$$\ldots\ldots\ldots.. (2)$$

But $\nabla^2 u = h(\xi,\eta)$ and $\nabla^2 G = \delta(\xi - x, \eta - y)$

in D. Thus, we have

$$\begin{aligned} &\iint_D [G(\xi,\eta;x,y)h(\xi,\eta) - u(\xi,\eta)\delta(\xi - x,\eta - y)]d\xi d\eta \\ &= \int_B \left[G(\xi,\eta;x,y)\frac{\partial u}{\partial n} - u(\xi,\eta)\frac{\partial G}{\partial n}\right]ds \end{aligned} \qquad \ldots\ldots\ldots\ldots (3)$$

Since $G = 0$ and $u = f$ on B, and noting that G is symmetric, it follows that

$$u(x,y) = \iint_D G(x,y;\xi,\eta)h(\xi,\eta)d\xi d\eta + \int_B f\frac{\partial G}{\partial n}ds$$

which is the solution given in Sec. 5.2.

As a specific example, consider the Dirichlet problem for a unit circle. Then

$$\begin{aligned} \nabla^2 g = g_{\xi\xi} + g_{\eta\eta} = 0 \quad & \text{in } D \\ g = -F \quad & \text{on } B \end{aligned} \quad \ldots\ldots\ldots\ldots (4)$$

But we already have from Eq. (5) that $F = (1/2\pi)\log r$.

If we introduce the polar coordinates (see Fig. 5.1) $\rho, \theta, \sigma, \beta$ by means of the equations

$$\begin{aligned} x = \rho\cos\theta, \quad & \xi\sigma\cos\beta \\ y = \rho\sin\theta, \quad & \eta = \sigma\sin\beta \end{aligned} \quad \ldots\ldots\ldots\ldots (5)$$

then the solution of Equation (4)

$$g = \frac{a_0}{2} + \sum_{n=1}^{\infty} \sigma^n (a_n \cos n\beta + b_n \sin n\beta)$$

Where $g = -\frac{1}{4\pi}\log[1 + \rho^2 - 2\rho\cos(\beta - \theta)]$ on B

By using the relation $\log[1 + \rho^2 - 2\rho\cos(\beta - \theta)] = -2\sum_{n=1}^{\infty} \frac{\rho^n \cos n(\beta - \theta)}{n}$

and equating the coefficients of $\sin n\beta$ and $\cos n\beta$ to determine a_n and b_n,

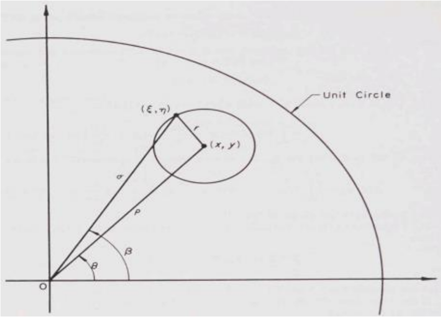

Figure 5.1

we find $\quad a_n = \frac{\rho^n}{2\pi n}\cos n\theta$

$\quad b_n = \frac{\rho^n}{2\pi n}\sin n\theta$

It therefore follows that

$$g(\rho, \theta; \sigma, \beta) = \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{(\sigma\rho)^n}{n}\cos n(\beta - \theta)$$

$$= -\frac{1}{4\pi}\log[1 + (\sigma\rho)^2 - 2(\sigma\rho)\cos(\beta - \theta)]$$

Hence the Green's function for the problem is

$$G(\rho,\theta;\alpha,\beta)=\frac{1}{4\pi}\log[\sigma^2+\rho^2-2\sigma\rho\cos(\beta-\theta)]$$
$$-\frac{1}{4\pi}\log[1+(\sigma\rho)^2-2\sigma\rho\cos(\beta-\theta)]$$

from which we find $\left.\frac{\partial G}{\partial n}\right|_{\text{on } B}=\left(\frac{\partial G}{\partial \sigma}\right)_{\sigma=1}=\frac{1}{2\pi}\frac{1-\rho^2}{[1+\rho^2-2\rho\cos(\beta-\theta)]}$

If $h=0$, then solution, sec 5.4 equation (3) reduces to the Poisson integral formula similar to sec 5.4 equation (10) and assumes the form $u(\rho,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\beta-\theta)}f(\beta)d\beta$

5.5. Dirichlet Problem for the Helmholtz Operator:

We will now determine the Green's function solution of the Dirichlet problem involving the Helmholtz operator, namely,

$$\begin{aligned}\nabla^2u+\kappa^2u&=h \quad \text{in } D\\ u&=f \quad \text{on } B\end{aligned} \quad \ldots\ldots\ldots\ (1)$$

where D is a circular domain of unit radius with boundary B. Then the Green's function must satisfy

$$\begin{aligned}\nabla^2G+\kappa^2G&=\delta(\xi-x,\eta-y) \quad \text{in } D\\ G&=0 \quad \text{on } B\end{aligned} \quad \ldots\ldots\ldots\ (2)$$

Again, we seek the solution in the form

$$G(\xi,\eta;x,y)=F(\xi,\eta;x,y)+g(\xi,\eta;x,y)$$

From Equation (6), we have $F=\frac{1}{4}Y_0(\kappa r)$ ....(3)

where $r=[(\xi-x)^2+(\eta-y)^2]^{\frac{1}{2}}$. The function g must satisfy

$$\begin{aligned}\nabla^2g+\kappa^2g&=0 \quad \text{in } D\\ g&=-\frac{1}{4}Y_0(\kappa r) \quad \text{on } B\end{aligned} \quad \ldots\ldots\ldots\ (4)$$

the solution of which can be easily determined by the method of separation of variables. Thus, the solution in the polar coordinates defined by sec 5. equation (5) may be written in the form

$$g(\rho,\theta;\sigma,\beta)=\sum_{n=0}^{\infty}J_n(\kappa\sigma)[a_n\cos n\beta+b_n\sin n\beta] \quad \ldots\ldots\ldots\ (5)$$

where

$$\left.\begin{aligned} a_0 &= -\frac{1}{8\pi J_0(\kappa)}\int_{-\pi}^{\pi} Y_0\left[\kappa\sqrt{1+\rho^2-2\rho\cos(\beta-\theta)}\right]d\beta \\ a_n &= -\frac{1}{4\pi J_n(\kappa)}\int_{-\pi}^{\pi} Y_0\left[\kappa\sqrt{1+\rho^2-2\rho\cos(\beta-\theta)}\right]\cos n\beta d\beta \\ b_n &= -\frac{1}{4\pi J_n(\kappa)}\int_{-\pi}^{\pi} Y_0\left[\kappa\sqrt{1+\rho^2-2\rho\cos(\beta-\theta)}\right]\sin n\beta d\beta \end{aligned}\right\} n = 1,2,\dots$$

To find the solution of the Dirichlet problem, we multiply both sides of the first equation of Equation (1) by G and integrate. Thus, we have

$$\iint_D (\nabla^2 u + \kappa^2 u)G(\xi,\eta;x,y)d\xi d\eta = \iint_D h(\xi,\eta)G(\xi,\eta;x,y)d\xi d\eta$$

We then apply Green's theorem on the left side of the preceding equation and obtain

$$\iint_D h(\xi,\eta)G(\xi,\eta;x,y)d\xi d\eta - \iint_D u(\nabla^2 G + \kappa^2 G)d\xi d\eta = \int_B (Gu_n - uG_n)ds$$

But $\nabla^2 G + \kappa^2 G = \delta(\xi - x, \eta - y)$ in D and $G = 0$ on B. We therefore have

$$u(x,y) = \iint_D h(\xi,\eta)G(\xi,\eta;x,y)d\xi d\eta + \int_B f(\xi,\eta)G_n ds$$

where G is given by Eqs. Sec 5.5 eqn (3) & (5).

5.6. Method of Images:

We shall describe another method of obtaining Green's function. This method, called the method of images, is based essentially on the construction of Green's function for a finite domain from that of an infinite domain. The disadvantage of this method is that it can be applied only to problems with simple boundary geometries.

As an illustration, we consider the same Dirichlet problem solved in Sec. 10.4.

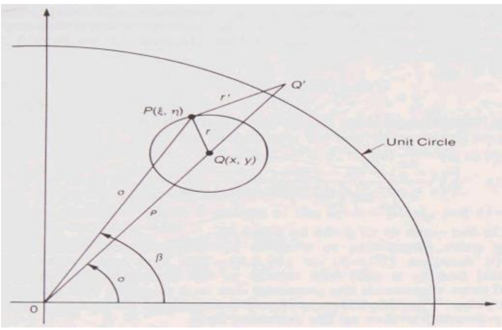

Figure 5.2

Let $P(\xi,\eta)$ be a point in the unit circle D, and let $Q(x,y)$ be the source point also in D. The distance between P and Q is r. Let Q' be the image which lies outside of D on the ray from the origin opposite to the source point Q (as shown in Fig 5.2) such that $OQ/\sigma = \sigma/OQ'$ where σ is the radius of the circle passing through P centered at the origin.

Since the two triangles OPQ and OPQ' are similar by virtue of the hypothesis $(OQ)(OQ') = \sigma^2$ and by possessing a common angle at O, we have $\frac{r'}{r} = \frac{\sigma}{\rho}$

where $r' = PQ'$ and $\rho = OQ$.

If $\sigma = 1$, Equation (1) becomes $\frac{r}{r'}\frac{1}{\rho} = 1$

Then we can clearly see that the quantity

$$\frac{1}{2\pi}\log\left(\frac{r}{r'}\frac{1}{\rho}\right) = \frac{1}{2\pi}\log r - \frac{1}{2\pi}\log r' + \frac{1}{2\pi}\log\frac{1}{\rho} \quad \ldots\ldots\ldots\ldots (2)$$

which vanishes on the boundary $\sigma = 1$, is harmonic in D except at Q,

$$G = \frac{1}{2\pi}\log r - \frac{1}{2\pi}\log r' + \frac{1}{2\pi}\log\frac{1}{\rho} \quad \ldots\ldots\ldots\ldots (3)$$

Noting that Q' is at $(1/\rho,\theta), G$ in polar coordinates takes the form

$$\begin{aligned} G(\rho,\theta;\sigma,\beta) =& \frac{1}{4\pi}\log[\sigma^2 + \rho^2 - 2\sigma\rho\cos(\beta-\theta)] \\ & -\frac{1}{4\pi}\log\left[\frac{1}{\sigma^2} + \rho^2 - 2\frac{\rho}{\sigma}\cos(\beta-\theta)\right] + \frac{1}{2\pi}\log\frac{1}{\sigma} \end{aligned} \quad \ldots\ldots\ldots\ldots (4)$$

The first term represents the potential due to a unit line charge at the source point, whereas the second term represents the potential due to a negative unit charge at the image point. The third term represents a uniform potential. The sum of these potentials makes up the potential field.

Example 1:

To illustrate an obvious and simple case, consider the semi-infinite plane $\eta > 0$. The problem is to solve

$$\nabla^2 u = h \text{ in } \eta > 0$$
$$u = f \text{ on } \eta = 0$$

The image point should be obvious by inspection. Thus, if we construct

$$G = \frac{1}{4\pi}\log[(\xi - x)^2 + (\eta - y)^2] - \frac{1}{4\pi}\log[(\xi - x)^2 + (\eta + y)^2]$$

the condition that $G = 0$ on $\eta = 0$ is clearly satisfied. It is also evident that G is harmonic in $\eta > 0$ except at the source point, and that G satisfies sec(5.2) equation (3).

With $G_n|_B = [-G_\eta]_{\eta=0}$, the sec(5.4) equation(3) is thus given by

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{(\xi - x)^2 + y^2}$$
$$+ \frac{1}{4\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} \log\left[\frac{(\xi - x)^2 + (\eta - y)^2}{(\xi - x)^2 + (\eta + y)^2}\right] h(\xi,\eta)d\xi d\eta$$

Example 2:

Another example that illustrates the method of images well is the Robin's problem on the quarter infinite plane, namely

$$\nabla^2 u = h(\xi,\eta) \qquad \text{in } \xi > 0,$$
$$u = f(\eta) \qquad \text{on } \xi = 0$$
$$u_n = g(\xi) \qquad \text{on } \eta = 0$$

This is illustrated in Fig. 5.3.

Let $(-x,y), (-x,-y)$, and $(x,-y)$ be the three image points of the source point (x,y). Then, by inspection, we can immediately construct Green's function

$$G = \frac{1}{4\pi}\log\frac{[(\xi - x)^2 + (\eta - y)^2][(\xi - x)^2 + (\eta + y)^2]}{[(\xi + x)^2 + (\eta - y)^2][(\xi + x)^2 + (\eta + y)^2]}$$

This function satisfies $\nabla^2 G = 0$ except at the source point, and $G = 0$ on $\xi = 0$ and $G_\eta = 0$ on $\eta = 0$.

The solution from Equation (2) is thus

$$u(x,y) = \iint_D G h d\xi d\eta + \int_B (G u_n - u G_n) ds$$
$$= \int_0^{\infty}\int_0^{\infty} G h d\xi d\eta + \int_0^{\infty} g(\xi) G(\xi, 0; x, y) d\xi$$
$$+ \int_0^{\infty} f(\eta) G_\xi(0,\eta; x, y) d\xi$$

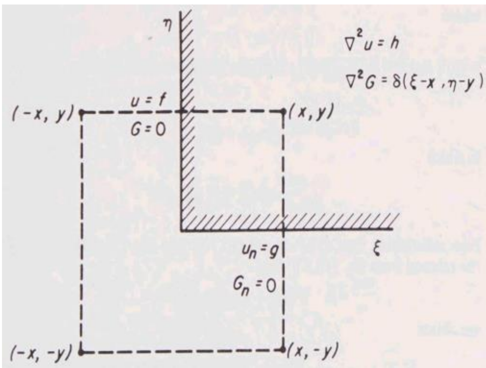

Figure 5.3

5.7. Method of Eigen Functions:

We consider the boundary value problem

$$\begin{aligned} \nabla^2 u &= h \quad \text{in } D \\ u &= f \quad \text{on } B \end{aligned} \quad \ldots\ldots\ldots\ldots (1)$$

For this problem, G must satisfy

$$\begin{aligned} \nabla^2 G &= \delta(\xi - x, \eta - y) \quad \text{in } D \\ G &= 0 \quad \text{on } B \end{aligned} \quad \ldots\ldots\ldots\ldots (2)$$

and hence the associated eigenvalue problem is

$$\begin{aligned} \nabla^2 \phi + \lambda\phi &= 0 \quad \text{in } D \\ \phi &= 0 \quad \text{on } B \end{aligned} \quad \ldots\ldots\ldots\ldots (3)$$

Let ϕ_{mn} be the eigenfunctions and λ_{mn} be the corresponding eigenvalues. We then expand G and δ in terms of the eigenfunctions ϕ_{mn}. Consequently, we write

$$G(\xi, \eta; x, y) = \sum_m \sum_n a_{mn}(x, y)\phi_{mn}(\xi, \eta) \ldots\ldots\ldots\ldots\ldots (4)$$

$$\delta(\xi - x, \eta - y) = \sum_m \sum_n b_{mn}(x, y)\phi_{mn}(\xi, \eta) \ldots\ldots\ldots\ldots\ldots (5)$$

where

$$\begin{aligned} b_{mn} &= \frac{1}{\|\phi_{mn}\|^2} \iint_D \delta(\xi - x, \eta - y)\phi_{mn}(\xi, \eta) d\xi d\eta \\ &= \frac{\phi_{mn}(x, y)}{\|\phi_{mn}\|^2} \quad \ldots\ldots\ldots\ldots\ldots (6) \end{aligned}$$

in which $\|\phi_{mn}\|^2 = \iint_D \phi_{mn}^2 d\xi d\eta$

Now substituting Equations (4) and (5) into equation (2) and using the relation from equation (3) that

$$\nabla^2 \phi_{mn} + \lambda_{mn}\phi_{mn} = 0$$

we obtain

$$-\sum_m \sum_n \lambda_{mn} a_{mn}(x, y)\phi_{mn}(\xi, \eta) = \sum_m \sum_n \frac{\phi_{mn}(x, y)\phi_{mn}(\xi, \eta)}{\|\phi_{mn}\|^2}$$

Hence $a_{mn}(x, y) = -\frac{\phi_{mn}(x,y)}{\lambda_{mn}\|\phi_{mn}\|^2}$

and the Green's function is therefore given by

$$G(\xi, \eta; x, y) = -\sum_m \sum_n \frac{\phi_{mn}(x, y)\phi_{mn}(\xi, \eta)}{\lambda_{mn}\|\phi_{mn}\|^2}$$

Example 1:

As a particular example, consider the Dirichlet problem in a rectangular domain

$$\nabla^2 u = h \qquad \text{in } D$$
$$u = 0 \qquad \text{on } B$$

The Eigen functions can be obtained explicitly by the method of separation of variables. We assume a solution in the form

$$u(\xi,\eta) = X(\xi)Y(\eta)$$

Substitution of this in

$$\nabla^2 u + \lambda u = 0 \qquad \text{in } D$$
$$u = 0 \qquad \text{on } B$$

yields, with α^2 as separation constant,

$$X'' + \alpha^2 X = 0$$
$$Y'' + (\lambda - \alpha^2)Y = 0$$

With the homogeneous boundary conditions $X(0) = X(a) = 0$ and $Y(0) = Y(b) = 0, X$ and Y are found to be

$$X_m(\xi) = A_m \sin\frac{m\pi\xi}{a}$$
$$Y_n(\eta) = B_n \sin\frac{n\pi\eta}{b}$$

We then have

$$\lambda_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \quad \text{with} \quad \alpha = \frac{m\pi}{a}$$

Thus, we obtain the Eigen functions

$$\phi_{mn}(\xi,\eta) = \sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$$

Knowing ϕ_{mn}, we compute $\|\phi_{mn}\|$ and obtain

$$\|\phi_{mn}\|^2 = \int_0^a \int_0^b \sin^2\frac{m\pi\xi}{a}\sin^2\frac{n\pi\eta}{b}\,d\xi d\eta$$
$$= \frac{ab}{4}$$

We thus obtain from Equation (8) the Green's function

$$G(\xi,\eta;x,y) = -\frac{4ab}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}}{(m^2b^2 + n^2a^2)}$$

5.8. Higher Dimensional Problem:

The Green's function method can be easily extended for applications in three and more dimensions. Since most of the problems encountered in the physical sciences are in three dimensions, we will illustrate some examples suitable for practical application.

First of all, let us extend our definition of Green's function in three dimensions.

The Green's function for the Dirichlet problem involving the Laplace operator is the function that satisfies

a. $\nabla^2 G = \delta(x-\xi, y-\eta, z-\zeta)$ in R .... (1)

$G = 0$ on S .... (2)

b. $G(x,y,z;\xi,\eta,\zeta) = G(\xi,\eta,\zeta;x,y,z)$ ....(3)

c. $\lim_{\varepsilon\to 0} \iint_{S_\varepsilon} \frac{\partial G}{\partial n} dS = 1$ ....(4)

where n is the outward unit normal to the surface

$S_\varepsilon: (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 = \varepsilon^2$

Proceeding as in the two-dimensional case, the solution of the Dirichlet problem

$\nabla^2 u = h$ in R

$u = f$ on S .... (5)

Is $u(x,y,z) = \iiint_R GhdR + \iint_S fG_n dS$ ....(6)

Again we let $G(\xi,\eta,\zeta;x,y,z) = F(\xi,\eta,\zeta;x,y,z) + g(\xi,\eta,\zeta;x,y,z)$

Where $\nabla^2 F = \delta(x-\xi, y-\eta, z-\zeta)$ in R

And $\nabla^2 g = 0$ in R

$g = -F$ on S

Example 1:

We consider a spherical domain with radius a. We must have $\nabla^2 F = 0$

except at the source point. For $r = [(\xi-x)^2 + (\eta-y)^2 + (\zeta-z)^2]^{\frac{1}{2}} > 0$

with (x,y,z) as the origin, we have

$$\nabla^2 F = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dF}{dr}\right) = 0$$

Integration then yields

$$F = A + \frac{B}{r} \text{ for } r > 0$$

Applying the condition (4) we obtain $\lim_{\varepsilon\to 0}\iint_{S_\varepsilon} G_n dS = \lim_{\varepsilon\to 0}\iint_{S_\varepsilon} F_r dS = 1$

Consequently, $B = -1/4\pi$ and A is arbitrary. If we set $A = 0$ for convenience, [16] we have

$F = -\frac{1}{4\pi r}$ .... (7)

We apply the method of images to obtain the Green's function. If we draw a three-dimensional diagram analogous to Fig. 1, we will have a relation similar to (1), namely,

$r' = \frac{a}{\rho} r$ .... (8)

where r' and ρ are measured in three-dimensional space. Thus, we seek Green's function

$G = \frac{-1}{4\pi r} + \frac{a/\rho}{4\pi r'}$ .... (9)

which is harmonic everywhere in r except at the source point, and is zero on the surface S.

In terms of spherical coordinates

$$\begin{aligned} \xi &= \tau\cos\psi\sin\alpha, & x &= \rho\cos\phi\sin\theta \\ \eta &= \tau\sin\psi\sin\alpha, & y &= \rho\sin\phi\sin\theta \\ \zeta &= \tau\cos\alpha, & z &= \rho\cos\theta \end{aligned}$$

G can be written in the form

$$G = \frac{-1}{4\pi(\tau^2+\rho^2-2\tau\rho\cos\gamma)^{\frac{1}{2}}} + \frac{1}{4\pi\left[\frac{\tau^2\rho^2}{a^2}+a^2-2\tau\rho\cos\gamma\right]^{\frac{1}{2}}} \quad \text{........... (10)}$$

where γ is the angle between r and r'. Now differentiating G, we have

$$\left[\frac{\partial G}{\partial \tau}\right]_{\tau=a} = \frac{a^2-\rho^2}{4\pi a(a^2+\rho^2-2a\rho\cos\gamma)^{\frac{3}{2}}}$$

Thus, the solution of the Dirichlet problem for $h = 0$ is

$$u(\rho,\theta,\phi) = \frac{a(a^2-\rho^2)}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{f(\alpha,\psi)\sin\alpha\, d\alpha\, d\psi}{(a^2+\rho^2-2a\rho\cos\gamma)^{\frac{3}{2}}} \quad \text{........... (11)}$$

where $\cos\gamma = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos(\psi-\phi)$. This integral is called the three-dimensional Poisson integral formula.

For the exterior problem where the outward normal is radially inward towards the origin, the solution can be simply obtained by replacing $(a^2-\rho^2)$ by (ρ^2-a^2) in Eqn (11).

Example 2:

Another example involving the Helmholtz operator is the three-dimensional radiation problem

$$\begin{aligned} \nabla^2 u + \kappa^2 u &= 0 \\ \lim_{r\to\infty} r(u_r + i\kappa u) &= 0 \end{aligned}$$

where $i = \sqrt{-1}$; the limit condition is called the radiation condition, and r is the field point distance.

In this case, the Green's function must satisfy

$\nabla^2 G + \kappa^2 G = \delta(\xi - x, \eta - y, \zeta - z)$

Since the point source solution is dependent only on r, we write the Helmholtz equation

$$G_{rr} + \frac{2}{r} G_r + \kappa^2 G = 0 \text{ for } r > 0$$

Note that the source point is taken as the origin. If we write the above equation in the form

$$(Gr)_{rr} + \kappa^2 (Gr) = 0 \text{ for } r > 0$$

then the solution can easily be seen to be $Gr = Ae^{i\kappa r} + Be^{-i\kappa r}$

or $G = A\frac{e^{i\kappa r}}{r} + B\frac{e^{-i\kappa r}}{r}$

In order for G to satisfy the radiation condition $\lim_{r\to\infty} r(G_r + i\kappa G) = 0$

$A = 0$ and G thus takes the form $G = B\frac{e^{-i\kappa r}}{r}$

To determine B we have $\lim_{\varepsilon\to 0} \iint_{S_\varepsilon} \frac{\partial G}{\partial n} dS = -\lim_{\varepsilon\to 0} \iint_{S_\varepsilon} B\frac{e^{-i\kappa r}}{r}\left(\frac{1}{r} + i\kappa\right) dS = 1$

from which we obtain $B = -1/4\pi$, and consequently, G becomes $G = -\frac{e^{-i\kappa r}}{4\pi r}$

Note that this reduces to $-1/4\pi r$ when $\kappa = 0$.

5.9. Neumann Problem:

We have noted in the chapter on boundary-value problems that the Neumann problem requires more attention than Dirichlet's problem, because an additional condition is necessary for the existence of a solution of the Neumann problem.

Let us now consider the Neumann problem

$$\nabla^2 u + \kappa^2 u = h \quad \text{in } R$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } S$$

By the divergence theorem, we have

$$\iiint_R \nabla^2 u dR = \iint_S \frac{\partial u}{\partial n} dS$$

Thus, if we integrate the Helmholtz equation and use the preceding result, we obtain

$$\kappa^2 \iiint_R u dR = \iiint_R h dR$$

In the case of Poisson's equation where $\kappa = 0$, this relation is satisfied only when

$$\iiint_R h dR = 0$$

If we consider a heat conduction problem, this condition may be interpreted as the requirement that the next generation of heat be zero. This is physically reasonable since the boundary is insulated in such a way that the net flux across it is zero.

If we define Green's function G, in this case, by

$$\nabla^2 G + \kappa^2 G = \delta(\xi - x, \eta - y, \zeta - z) \qquad \text{in } R$$

$$\frac{\partial G}{\partial n} = 0 \qquad \text{on } S$$

Then we must have

$$\kappa^2 \iiint_R G dR = 1$$

which cannot be satisfied for $\kappa = 0$. But, we know by physical reasoning that a solution exists if

$$\iiint_R h dR = 0$$

Hence we will modify the definition of Green's function so that

$$\frac{\partial G}{\partial n} = C \quad \text{on } S$$

where C is a constant. When integrating $\nabla^2 G = \delta$ over R, we obtain

$$C \iint_S dS = 1$$

It is not difficult to show that G remains symmetric if

$$\iint_S G dS = 0$$

Thus, under this condition, if we take C to be the reciprocal of the surface area, the solution of the Neumann problem for Poisson's equation is

$$u(x, y, z) = C^* + \iiint_R G(x, y, z; \xi, \eta, \zeta) h(\xi, \eta, \zeta) d\xi d\eta d\zeta$$

where C^* is a constant.

We should remark here that the method of Green's functions provides the solution in integral form. This is made possible by replacing a problem involving nonhomogeneous boundary conditions with a problem of finding Green's function G with homogeneous boundary conditions.

Regardless of methods employed, the Green's function of a problem with a nonhomogeneous equation and homogeneous boundary conditions is the same as the Green's function of a problem with a homogeneous equation and nonhomogeneous boundary conditions, since one problem can be transferred to the other without difficulty. To illustrate, consider the problem

$Lu = f$ in R
$u = 0$ on ∂R

where ∂R denotes the boundary of R.

If we let $v = w - u$, where w satisfies $Lw = f$ in R, then the problem becomes

$Lv = 0$ in R
$v = w$ on ∂R

Conversely, if we consider the problem

$Lu = 0$ in R
$u = g$ on ∂R

we can easily transform this problem into

$Lv = Lw \equiv w^*$ in R
$v = 0$ on ∂R

by putting $v = w - u$ and finding w that satisfies $w = g$ on ∂R.

In fact, if we have

$Lu = f$ in R
$u = g$ on ∂R

we can transform this problem into either one of the above problems.

Exercises:

1.Prove that the Green's function for a region, if it exists, is unique.

2. Determine the Green's function for the exterior Dirichlet problem for a unit circle.

$\nabla^2 u = 0 \; in \; r > 1 \;, u = f \; in \; r = 1$

3.Determine the Green's function for the semi-infinite region $\xi > 0$ for

$\nabla^2 G + \mathrm{k}^2 G = \delta\,(\xi - x, \eta - y, \zeta - z)$ G=0 on $\zeta = 0$.

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